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SURFACE WAVES IN WATER OF VARIABLE DEPTH*

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Introduction. The problem of irrotational gravity waves in water is, from the mathematical point of view, a problem in potential theory which involves a nonlinear boundary condition at the free surface. In addition, the shape of the free surface is itself not given a priori but is to be determined as part of the solution along with other quantities, such as the distribution of the velocity and the pressure. Very few successful attacks on the problem in this formulation have been made; among these are the proofs of the existence of steady periodic waves of special type by Levi-Civita [5],** and Struik [11], and a partial treatment of the problem of the so-called solitary wave by A. Weinstein [13].

Most of the literature on the subject of surface gravity waves is concerned with theories which result from the general nonlinear theory when simplifying assumptions of one kind or another are made. The present paper is concerned with two such theories:

1) Perhaps the best known and most extensively studied theory is that which results from the general theory when it is assumed that the amplitude of the waves at the surface and the velocity of the particles there are small enough that the free surface condition can be simplified by dropping the nonlinear terms; in addition, this condition may be prescribed at the original undisturbed surface of the water. The result is a problem in potential theory with a linear boundary condition of the mixed type. The greater part of our work here makes use of this theory. We shall refer to it as the *exact linear theory*, or simply as the *exact theory*.

2) The second theory furnishes an approximation to the exact linear theory which is based on the assumption that the depth of the water is small. The resulting theory yields a differential equation for the surface elevation of the water which turns out to be the wave equation. This approximate theory, which is often called the shallow water theory is used, for example, in discussing the tides in the ocean. In Sec. 7 we give a brief derivation of the shallow water theory which brings out the role of the depth of the water as the determining factor in the accuracy of the approximation. The usual derivation of the theory based on assuming that the pressure in the water is given by the same law as in hydrostatics does not bring this point out clearly. The

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** Numbers in square brackets refer to the bibliography at the end of the paper.

approach to the theory through the hydrostatic pressure relation also does not lend itself easily to generalization to other cases, such as the derivation of the shallow water theory when floating bodies are present.

Our principal object in this paper is to solve the problem of determining progressing waves over a uniformly sloping bottom by making use of the exact linear theory, to discuss the solutions numerically, and to compare them with the approximate solutions furnished by the shallow water theory.

Solutions for waves on sloping beaches in terms of the exact theory have been obtained by Hanson [3], Bondi [2], Miche [9] and H. Lewy [6]. The first author obtained one type of standing wave solution. The second and third authors obtained two types of standing wave solutions for the case of motion in two dimensions from which progressing wave solutions can be constructed. The first three writers, as well as the writer of the present paper, are concerned only with cases in which the bottom slopes at the special angles $\pi/2n$, with n an integer. The method employed in the present paper is different from those of the first three authors; in particular, the progressing wave solutions are obtained here in a closed form which lends itself well to detailed discussion. Also, the method employed in the present paper yields three-dimensional progressing wave solutions, that is progressing waves which approach an arbitrary plane wave at infinity (cf. Sec. 9).

The investigations which led to the present paper were begun in collaboration with H. Lewy, who then later extended the method to two-dimensional motions (cf. [6]) for a bottom sloping at the angles $p\pi/2n$ in which p is any odd integer and n is any integer such that $2n>p$. (The cases $p\neq 1$ are very much more complicated than those for $p=1$, by the way.)

The basic idea of the method devised by H. Lewy and used by him and the author is to obtain a differential equation for the desired velocity potential, through use of the boundary conditions, which is *not* the potential equation and which, as it turns out, permits an explicit integration. In the case of two-dimensional motions the problems can be treated, of course, by making use of analytic functions of a complex variable; in this case Lewy's differential equation becomes an ordinary non-homogeneous differential equation with constant coefficients for the complex potential, and this equation can be integrated to yield the desired solutions. In the three dimensional cases the solutions can also be obtained, as explained in Sec. 9, but the results are more complicated and more difficult to handle numerically. It may be of interest to observe that the method developed here for the mixed boundary value problem in wedge-shaped regions of angle $\pi/2n$ is not confined in its usefulness to solutions of the potential equation—it could also be extended to other linear partial differential equations.

In Sec. 1 the exact linear theory (which apparently goes back to Poisson) is formulated briefly. In Sec. 2 the theory is applied to yield the well known solutions for steady progressing waves in water of infinite depth. In addition, we show in Sec. 2 that these solutions are uniquely determined if the amplitude and velocity of the waves is bounded at ∞.[1] The method used to obtain this result is essentially the same as that employed later to obtain progressing wave solutions over a sloping bottom.

[1] The author was led to these rather general conditions guaranteeing the uniqueness of the solutions as the result of a conversation with A. Weinstein, who had previously obtained the same result by quite different methods for water of finite and constant depth (cf. [12]).

In Sec. 3 convenient dimensionless independent variables are introduced, and these variables are used in the remainder of the paper except in Sections 7 and 8.

In Sec. 4 the case of progressing waves coming from infinity in an ocean of infinite depth but bounded on one side by a vertical cliff is treated in considerable detail. Only the two dimensional case is considered—that is, the wave crests and all other curves of constant phase are assumed to be horizontal straight lines parallel to the cliff. Since the problem is then a potential problem in two dimensions, it is convenient to solve it in terms of analytic functions of a complex variable. In Sec. 5 the method used for the case of a vertical cliff is generalized to yield solutions in water over a plane bottom sloping at any angle $\pi/2n$, with n an integer. Again only the two dimensional case is treated. The essential step in the generalization requires the derivation of Lewy's differential equation, which turns out to be a differential equation of order n for an angle $\pi/2n$. The solutions of the differential equation which satisfy the boundary and regularity conditions are then given in this section. The solutions obtained are shown to be uniquely determined if they have at most a logarithmic singularity at the origin (that is, at the shore line) and satisfy certain boundedness conditions at ∞.[2] The method of determining the arbitrary constants in such a way that the boundary conditions are satisfied is discussed in Appendix I. In Appendix II, the behavior of the solutions at ∞ is investigated; we find that the solutions behave as one would expect, i.e. that they tend to the simple standing wave solutions for water of infinite depth. It is then readily seen that standing waves of arbitrary phase and amplitude at ∞ can be constructed from our solutions. In this way the existence of a unique set of standing wave solutions is established. For the angles $\pi/2n$, then, the problem of progressing waves has been solved completely for the two dimensional case.

In Sec. 6 the theory of Sections 4 and 5 is applied to the cases $n=1$, $n=2$, and $n=15$, i.e. to the cases of bottom slopes of 90°, 45°, and 6° respectively.[3] The standing wave solutions are given numerically in the form of graphs for a distance of a few wave lengths from shore. The numerical evaluation requires the calculation of the values of complex integrals of the form

$$E(z) = e^z \int_z^{+\infty} t^{-1} e^{-t}\, dt$$

taken over appropritate paths in the t-plane. A table of values of $E(z)$ for the range of values of interest to us was computed and is included here as an Appendix. This table was based on a previous table calculated by the Mathematical Tables Project [8]. It might be noted that the calculations for the case of the 6° slope were very laborious.

In all cases (except that of water of infinite depth everywhere) there are two types of standing waves: the one type has a finite amplitude all the way to shore, the other type has amplitudes which become logarithmically infinite as the shore is approached. At infinity the amplitude and wave length may be arbitrarily prescribed for both types, but the wave length at ∞ and the frequency are connected by the same relation as in water which is infinite in depth everywhere. These two types of standing

[2] This result (which is not obtained in the papers by Miche, Bondi, and Lewy cited above) means that no non-trivial solution exists which dies out at ∞.

[3] The paper of Miche [9], which gives the solutions in the general case, contains graphs for the 45° and 30° cases as well as approximate solutions for small angles of slope.

waves are always out of phase at ∞, but the ones remaining finite (for any fixed bottom slope) at the shore all have the same phase at infinity. (The solutions obtained by Hanson [3] are those which remain finite at the shore.) Thus all progressing waves furnished by the exact theory necessarily have large amplitudes near shore.

In all three cases treated numerically, i.e. those with 90°, 45°, and 6° slopes, the most striking general result is the following: The wave lengths and amplitudes change very little from the values at ∞ until points about a wave length from shore (wave length at ∞ is meant) are reached. Closer in shore the amplitude of any progressing wave becomes large. It is curious, however, that the amplitude (that is, the relative maximum or minimum) of a progressing wave at a point a wave length or two from shore can actually be about 10 per cent *less* than the value at ∞. This was found in all three cases, including the 6° case.

In Sec. 7 the shallow water theory is derived. In this theory, the wave amplitudes for a uniform bottom slope die out at ∞ like $x^{-1/4}$ (where x denotes distance from shore) while the wave lengths (or, rather, the distances between successive nodes) increase without limit. Thus if the wave lengths and amplitudes were to be taken with the same value at a given point some distance off shore in the two theories (i.e. the exact theory and the shallow water theory) the amplitude near shore as given by the shallow water theory would be much too high while at great distances from shore it would be too low.

In Sec. 8 the results of the exact theory are compared with those of the shallow water theory for the case of water of finite and constant depth in order to bring out the known fact that the accuracy of the shallow water theory for a simple progressing wave depends upon having the ratio of depth to wave length sufficiently small.

Progressing waves moving toward shore as given by both theories for the case of the 6° beach are compared graphically in Sec. 8 assuming the same frequency in both cases and also the same phase and amplitude at a point two or three wave lengths from shore.The results from there on toward shore do not differ greatly except at points very close to shore. This was to be expected, since the depth of the water at the point where the phase and amplitude are the same is about one-eighth of the wave length and hence one might expect the shallow water theory to be quite accurate. However, if the amplitudes had been made equal at a point 15 or 20 wave lengths from shore, the amplitudes given by the shallow water theory three wave lengths from shore would be 50 per cent to 60 per cent higher than those furnished by the exact theory. In other words, the amplitude variation with decrease in depth cannot be correctly estimated over distances of many wave lengths from a given point using the inverse fourth root law unless the wave length at the point is eight or ten times the depth of the water. On the other hand, as the results of the exact theory indicate, if the depth remains an appreciable fraction of the wave length the amplitude changes very little with changes in depth. We draw this conclusion from the fact that the wave amplitudes as given by the exact theory approach their values at ∞ very quickly when the depth approaches say half the wave length.

Another feature of our numerical results which is of interest concerns the variation in the phase or propagation velocity c of a progressing wave when the depth h varies. In Sec. 8 graphs are given which show the results of calculations of the phase velocity for the 6° case, using both the exact theory and the shallow water theory. In the case of the shallow water theory, it is found that the phase velocity c is practically

identical with that given by the formula $c=(gh)^{1/2}$. In the case of the exact theory, the phase velocity is given very accurately by the formula $c=[(\lambda g/2\pi)\tanh(2\pi h/\lambda)]^{1/2}$, that is, in both cases c is given by the formula which is exact (for that theory) only when the depth h is constant. (In the second formula λ is taken as the wave length at ∞.) In other words, the phase velocity at any point is given very accurately by the formula which is exact only for water of uniform depth equal to that at the point in question and for a steady progressing wave traveling in one direction only. At the same time, the results for the phase velocity as furnished by the two theories agree very well with each other for a distance of about three wave lengths from shore but from then on out the shallow water theory furnishes a value which is too high by an amount which increases without limit with increasing distance from shore. These remarks, it should be recalled, result from our calculations for a slope of 6°. With decreasing slope, it seems certain that the shallow water theory would be accurate up to distances comprising more and more wave lengths from shore.

All progressing wave solutions discussed above were obtained on the assumption that the wave comes from ∞ toward shore with no component which goes outward at ∞. Once the frequency and amplitude at ∞ are prescribed, the additional condition that the wave at ∞ is a progressing wave moving toward shore leads to a unique solution, which, as we have already mentioned, has a logarithmic singularity at the shore line. The solution is also uniquely determined if the singularity at the shore line is prescribed—the behavior at ∞ is then determined. Our theory thus furnishes us with two types of standing wave solutions from which solutions behaving like arbitrary simple harmonic progressing waves at ∞ can be constructed, but it furnishes no criterion by which one can decide what type of wave would actually occur in practice. Our assumption, in the numerical cases treated, that the waves move from ∞ toward shore with no reflection from the shore back to ∞ was the result of information on the phase velocities as measured on beaches with small slopes; these measurements agree rather well with the theoretical results discussed in the preceding paragraph for a progressing wave moving toward shore. The physical mechanism which prevents the reflection of waves from the shore can be understood as the result of the partial loss of energy from turbulence and the conversion of the remainder into an undertow through the occurrence of breakers. If, however, the slope of the beach is large it may well be that a standing wave, denoting perfect reflection, could occur.

In Sec. 9 we solve the problem of progressing waves in an ocean of infinite depth bounded on one side by a vertical cliff when the wave crests are not assumed to be parallel to the shore line (as in Sec. 4); that is, we solve a three-dimensional problem using the exact theory. Solutions are obtained which tend at ∞ to an arbitrary plane wave. In all of the solutions obtained in the preceding sections by means of the exact theory the discussion was greatly facilitated by the use of analytic functions of a complex variable. In the present three-dimensional case this approach is no longer possible. Nevertheless, the process of obtaining the solutions remains analogous to that using complex functions. The solution for the case of a vertical cliff only is obtained, but it is readily seen how three-dimensional progressing wave solutions for slopes of angles $\pi/2n$ could also be found. The solution for the case of the vertical cliff is also evaluated numerically in Sec. 9 for the case of a progressing wave with wave crests tending to a straight line at ∞ which makes an angle of 30° with the shore line. One of the figures given in Sec. 9 shows the contours of the wave surface. In this case,

as well as in the previous two-dimensional cases, it turns out that there is a point near the cliff where the wave crests are lower than they are at ∞, although the elevation of the wave crests becomes infinite upon approaching the cliff.

Finally, the author takes pleasure in acknowledging the help and advice he received from a number of his colleagues and co-workers. The actual solution of Lewy's differential equation and the determination of the constants to satisfy the boundary conditions—no small task in itself—was carried out by E. Bromberg and E. Isaacson. The extensive numerical computations were completed by E. Isaacson, B. Grossmann, and J. Butler.

1. Resumé of general theory of surface waves of small amplitude. In this section we state briefly the well-known mathematical formulation of the problem of surface waves of small amplitude in water. (See, for example, Lamb: Hydrodynamics, Chap IX; or Milne-Thompson: Theoretical Hydrodynamics, Chap. XIV.) The water is assumed to fill the region $-h(x, z) \leqq y \leqq 0$ when at rest. The non-negative quantity h is the (variable) depth of the water. The motion is assumed to be irrotational, so that a velocity potential $\Phi(x, y, z; t)$ exists, in which Φ depends not only on x, y, z but also on the time t. Hence Φ satisfies the Laplace equation

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0. \tag{1.1}$$

A solution of this differential equation in the region $-h \leqq y \leqq 0$ is to be found which satisfies appropriate boundary conditions. The condition to be satisfied at the free surface $y=0$ is

$$\frac{\partial^2\Phi}{\partial t^2} + g\,\frac{\partial\Phi}{\partial y} = 0, \tag{1.2}$$

which results from the Bernoulli law and the assumption that nonlinear terms in the displacement and velocity of the free surface can be neglected. At the bottom $y = -h(x, z)$ we require of course that the derivative of Φ normal to the bottom surface should vanish:

$$\left.\frac{\partial\Phi}{\partial n}\right|_{y=-h} = 0. \tag{1.3}$$

We shall be interested solely in phenomena which are periodic in the time t. It is therefore convenient to replace Φ in the above equation by $e^{i\sigma t}\varphi$, in which φ depends only on x, y, z and not on t.

The conditions on φ are the same as those on Φ except that (1.2) becomes

$$g\,\frac{\partial\varphi}{\partial y} - \sigma^2\varphi = 0. \tag{1.2$'$}$$

In addition to the differential equation (1.1) and the boundary conditions (1.2′) and (1.3), the functions $\varphi(x, y, z)$ should be required to satisfy certain conditions at infinity which lead to unique solutions of the type desired on physical grounds. These conditions will be formulated in the following sections for the special cases of interest to us.

Once φ has been determined, the vertical displacement $\eta(x, z; t)$ of the free surface is determined (see the books of Lamb and Milne-Thompson cited above) by the formula

$$\eta = \frac{1}{g}\frac{\partial}{\partial t}(e^{i\sigma t}\varphi)\Big|_{y=0}. \tag{1.4}$$

2. Plane traveling waves in water of infinite depth. We seek solutions of the boundary value problem partially formulated in equations (1.1), (1.2′) and (1.3) which have the form of plane traveling waves for the case of infinite depth of water. The potential function φ may be assumed to depend only upon x and y and not on z: $\varphi=\varphi(x, y)$, and it is to be determined in the entire half plane $y\leqq 0$.

The functions

$$\varphi = e^{my}\cos(mx+\alpha), \text{ with } m \text{ and } \alpha \text{ arbitrary}, \tag{2.1}$$

are familiar potential functions which obviously yield "standing waves" upon reintroduction of the time factor.[4] The amplitude of these waves decreases exponentially with the depth. The condition (1.2′) at the free surface is satisfied if the following relation holds:

$$\sigma^2 = gm = g\cdot\frac{2\pi}{\lambda}, \tag{2.2}$$

in which λ is the wave length. This yields, of course, a relation between the frequency and wave length which characterizes the type of dispersion encountered with surface gravity waves in water of infinite depth.

Since our problem is linear and homogeneous, we may take linear combinations of the standing wave solutions to obtain "traveling wave" solutions given by the velocity potential $\Phi(x, y, z, t)$ as follows:

$$\Phi = e^{my}\cos(mx+\sigma t+\alpha), \tag{2.3}$$

in which α is an arbitrary phase shift. This would represent a wave traveling in the direction of the negative x-axis. Of course, the relation (2.2) between wave length and frequency must always be satisfied. The above theory is the well-known classical theory, which is due to Poisson.

A point which seems not to have been raised in the standard treatises is that concerning the uniqueness of the solutions given by (2.1). It is of interest to deal with this question here since the reasoning used is later on generalized in such a way as to yield the solutions for the problem of waves in water over a uniformly sloping bottom.

We wish to show the following: If φ is a regular potential function in the half-plane $y\leqq 0$ which satisfies the free surface condition $\sigma^2\varphi=g\partial\varphi/\partial y$ on $y=0$ and if the function $\varphi(x, y)$ and its first derivatives φ_x and φ_y are uniformly bounded in (x, y) as $x^2+y^2\to\infty$, then $\varphi\equiv Ae^{my}\cos(mx+\alpha)$, that is, if the velocity and the vertical displacement of the water are bounded at ∞, then $\varphi(x, y)$ is either identically zero or it is of the type (2.1) with $A\neq 0$. The mixed boundary condition at the free surface in conjunction with the relatively weak condition at ∞ thus leads to this rather narrow class of solutions.[5]

Our proof of this theorm requires the introduction of the analytic function

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[4] It is a curious fact that this theory, which deals with what is perhaps the most familiar of all wave motions in nature, is governed by the potential equation rather than the wave equation. Nevertheless, we shall use the terms *wave-length, amplitude, phase*, etc. with a meaning which is obvious from the context.

[5] The author's attention was called to this possibility by A. Weinstein, who had already proved the same theorem for the case in which the fluid has a finite and constant depth. See [12].

$f(z)=\varphi+i\psi$ of the complex variable[6] $z=x+iy$, the real part of which is the potential function φ. It is convenient at this point to translate our conditions at ∞ on $\varphi(x, y)$ into conditions on $f(z)$ at ∞. In both cases, of course, these functions are defined in the half-plane $y\leqq 0$. On account of the Cauchy-Riemann equations it follows at once that $|df/dz|$ is not greater than $|\varphi_x|+|\varphi_y|$ and thus is uniformly bounded at ∞ since this is the case for $|\varphi_x|$ and $|\varphi_y|$. Since $f(z)=\int^z f'(\xi)d\xi$ it is clear that we have

$$|f(z)| < M|z|,$$

where M a positive constant, for all sufficiently large $|z|$.

In addition, it is readily seen that $|df/dz|$ tends to zero when $|z|\to\infty$ along any ray in $y\leqq 0$ which is not parallel to the real axis. Since we wish to make use of a slightly more general result later on it is convenient to formulate here the following *Lemma: If $f(z)=\varphi+i\psi$ is analytic and regular in the interior of a sector of the complex plane and φ is uniformly bounded at ∞, the absolute value of any derivative of $f(z)$ tends to zero when $|z|\to\infty$ along any ray which is not parallel to either of the boundary rays of the sector.* The lemma is an almost immediate consequence of the assumption that φ is uniformly bounded at ∞. We consider $\varphi(x, y)$ to be expressed in terms of its values on the circumference of a circle with center at (x, y) and radius R through use of the Poisson integral formula. By differentiating both sides of this relation one easily obtains bounds for φ_x and φ_y at the point (x, y) of the form $|\varphi_x| <2M_1/R$, $|\varphi_y| <2M_1/R$ with M_1 a constant which may be taken as the maximum of $|\varphi|$ on the circle of radius R. It therefore follows that $|df/dz|_{z=x+iy}<4M_1/R$ in view of the Cauchy-Riemann equations. We now assume that M_1 is a fixed upper bound for φ for all sufficiently large z—that such a bound exists was assumed. As $|z|\to\infty$ along any ray not parallel to the sides of our sector, it is clear that $|df/dz|\to 0$ since we may allow R to tend to infinity (i.e. our domain will accommodate circles of arbitrarily large radius with centers on such a ray) while M_1 is fixed. Since φ_x and φ_y are potential functions which are well known to be bounded at ∞ in any closed sub-domain of the sector[7] it follows that the second derivatives of φ tend to zero and therefore also $|d^2f/dz^2|$ tends to zero along the rays in question. That the higher derivatives of $f(z)$ behave in the same manner is now obvious.

The condition on φ at the free surface leads to the following condition on $f(z)=\varphi+i\psi$ on the real axis. We may write successively

$$\left(g\frac{\partial}{\partial y}-\sigma^2\right)\varphi = \Re e\left(g\frac{\partial}{\partial y}-\sigma^2\right)(\varphi+i\psi)$$
$$= \Re e\left(gi\frac{d}{dz}-\sigma^2\right)f(z).$$

($\Re e$ means "the real part of" the expression which follows) the last step being a consequence of the fact that $f(z)$ is analytic. Thus the free surface condition may be written

$$\Re e\left(gi\frac{d}{dz}-\sigma^2\right)f = 0 \quad \text{for } z \text{ real.} \tag{2.4}$$

[6] In all but the final section of this paper we deal with two-dimensional problems only so that no confusion between the complex variable z and the third space variable should arise, particularly since the complex variable is not used in the final section.

[7] The bounds $2M_1/R$ for $|\varphi_x|$ and $|\varphi_y|$ obtained above could be used to yield this result.

We next introduce the analytic function $F(z)$ defined by

$$F(z) = \left(gi \frac{d}{dz} - \sigma^2\right) f(z). \tag{2.5}$$

Clearly $F(z)$ is regular in the lower half plane. Since $f(z)$ satisfies (2.4) it follows that the real part of $F(z)$ is zero on the real axis, and hence $F(z)$ can be continued analytically by reflection into the entire upper half-plane; thus the resulting function is regular in the entire plane, and any bounds for $|F(z)|$ in the lower half plane also hold in the upper half plane. From (2.5) and our conditions on $f(z)$ at ∞ it is clear that $|F(z)| < M_2|z|$ for all sufficiently large $|z|$ in the lower half plane, and hence also in the entire plane. It follows that $F(z)$ is linear, by application of Liouville's theorem to the function $[F(z) - F(0)]/z$; that is, it can be written as $F(z) = c_1z + c_2$. If we now introduce $c_1z + c_2$ for $F(z)$ in (2.5) and integrate to obtain $f(z)$ the result is

$$f(z) = Ae^{-imz} + Bz + C, \tag{2.6}$$

with $m = \sigma^2/g$. The constant A is an arbitrary complex constant. The constant B must, however, be zero since $|df/dz|$ tends to zero along all rays not parallel to the real axis by the Lemma proved above,[8] and the same is true of $|de^{-imz}/dz|$ since m is real and $\Re e(-imz) \to -\infty$ along such rays. The constant C must be pure imaginary because of the boundary condition (2.4), as one readily sees. Thus the only non-vanishing potential functions φ for which the velocity and surface elevation are bounded at ∞ are of the form (2.1).

3. Introduction of dimensionless quantities. In dealing with surface waves in the remainder of this paper it is convenient to work with dimensionless space variables x_1 and y_1 defined by $x_1 = mx$, $y_1 = my$, in which m is given by

$$\sigma^2 = gm = g \cdot \frac{2\pi}{\lambda}, \tag{3.1}$$

with σ the circular frequency and λ as wave length. We also replace the time variable t by a new variable t_1 given by $t_1 = \sigma t$. In these variables the surface condition (1.2′) is readily seen to take the form

$$\frac{\partial \varphi}{\partial y_1} - \varphi = 0 \quad \text{for} \quad y_1 = 0, \tag{3.2}$$

where φ of course satisfies $\nabla^2\varphi = 0$ in the new variables. However, since nearly all of our work from now on is carried out in the new variables, *we shall drop the subscripts but retain the surface condition in the form* (3.2). The original variables can always be reintroduced by replacing x and y in all of our results by mx and my and t by σt. This means that the reciprocal of the quantity m defined by (3.1) is our unit of length. In the course of our discussion on waves over a sloping bottom it will be shown that the relation (cf. (2.2)) $\sigma^2 = gm = 2\pi g/\lambda$ between frequency and wave length for water of infinite depth holds asymptotically as the depth of the water becomes infinite, when λ is the "wave length at ∞." Thus our unit of length in these cases is proportional to the wave length at ∞.

[8] At this point we use the assumption that φ is bounded.

The standing wave solutions Φ corresponding to (2.1) are, in the new variables (after dropping subscripts):

$$\Phi = e^{it}e^{y}\cos(x+\alpha). \tag{3.3}$$

4. Plane traveling waves in an ocean of infinite depth bounded on one side by a vertical cliff. As stated in the introduction, the main purpose of this paper is to study plane traveling waves in an ocean with a uniformly sloping bottom. In this section we deal in detail with the special case in which the "bottom" is vertical. Most of the essentials of the method to be employed in the more general cases are well illustrated in this case, while the formal apparatus necessary is much simpler than that needed for the general case. In our treatment of the more general cases we shall then feel free to condense the presentation in many particulars.

We assume that all quantities depend upon x and y only so that all curves of constant phase (the loci of the wave crests, for example) are parallel to the line of intersection of the free surface and the cliff forming the vertical boundary of the water.

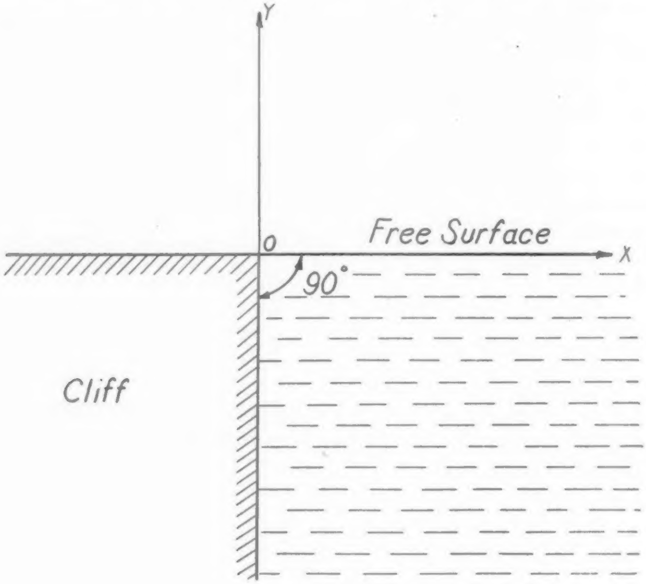

FIG. 1.

Thus we seek a potential function $\varphi(x, y)$ in the shaded area of Fig. 1 which satisfies the surface condition (see Sec. 3):

$$\frac{\partial\varphi}{\partial y} - \varphi = 0 \quad \text{when} \quad y = 0,\ x > 0 \tag{4.1}$$

and the condition at the vertical wall

$$\frac{\partial\varphi}{\partial x} = 0 \quad \text{when} \quad x = 0,\ y < 0. \tag{4.2}$$

As we have already stated, our purpose is to obtain potential functions φ which

satisfy (4.1) and (4.2) and which behave at ∞ like traveling waves moving toward shore. It seems reasonable to expect that a velocity potential $\Phi(x, y, t)$ which behaves at ∞ like the solution $e^{it}e^y \cos(x+\alpha)$ for water of infinite depth everywhere (cf. (3.3)) will exist in the present case. Or, as we could also put it, we may expect that two potential functions φ_1, φ_2 can be found which behave on the surface at ∞ like $\sin x$ and $\cos x$ respectively, since a reintroduction of the time factor e^{it} would yield two "standing wave solutions" which could be combined linearly (since our problem is linear and homogeneous) to yield a solution behaving like a traveling wave at ∞.[9] In what follows we shall obtain two such potential functions which are 90° "out of phase" at ∞. One such solution which is bounded and regular can be obtained immediately: The boundary condition (4.2) permits an analytic continuation of φ by reflection in the negative y axis into the entire half plane $y \leqq 0$, and we have already obtained solutions for this case in Sec. 2. Since only an even function of x is in question it follows that $\varphi_1 = Ae^y \cos x$ is the only solution regular in the entire fourth quadrant which, together with its first derivatives, is uniformly bounded at ∞ in this quadrant, because of the fact that the solutions obtained in Sec. 2 were shown to be unique under these circumstances. (It is clear that bounds for φ at ∞ in the fourth quadrant hold, after reflection, in the half plane $y \leqq 0$.) In other words, all non-singular solutions which are bounded at ∞ have the same phase at ∞, i.e. they behave like $\cos x$ there.[10] To obtain solutions "out of phase" with $e^y \cos x$ at ∞ it is therefore essential to admit a singularity. On the other hand, it is rather natural on physical grounds to expect a singularity at the origin (i.e. at the water line on the vertical cliff) of the type of a source or sink if a progressing wave coming toward shore from ∞ occurs.[11] This point has already been discussed in the introduction.

We are now in a position to complete the formulation of our boundary value problem by prescribing conditions on φ at the origin and at ∞. We require, in accordance with the remarks above, that φ should possess a representation of the form

$$\varphi = \bar{\varphi} \log r + \bar{\bar{\varphi}} \tag{4.3$_0$}$$

valid near the origin, with $r = x^2 + y^2$ and $\bar{\varphi}$ and $\bar{\bar{\varphi}}$ functions which together with their first two derivatives are bounded in a neighborhood of the origin. At ∞ we require that φ and its partial derivatives of the first two orders be uniformly bounded, i.e. that a constant M exists such that

$$|\varphi| + \sum |\varphi^{(n)}| < M, \tag{4.3$_\infty$}$$

for all sufficiently large $x^2 + y^2$, in which the sum is taken over all first and second partial derivatives of φ.[12] The conditions (4.1), (4.2), (4.3$_0$) and (4.3$_\infty$) constitute the

[9] Of course the motion will not be a steady wave motion in general, but one which "approaches" a steady motion at ∞.

[10] Upon reintroduction of the original space variables it is seen that this type of solution includes waves of all possible wave lengths.

[11] The paper of Hanson [3] mentioned in the introduction contains only the regular solution. It might be of interest to note that the starting point of the present investigation was the conjecture that solutions out of phase with those of Hanson at ∞ could be obtained by admitting a singularity corresponding to a source or sink at the origin.

[12] These requirements are more stringent than would be necessary to ensure the existence and uniqueness of the type of solutions desired. However, we are not interested in this paper in formulating conditions at ∞ and at the origin in the most general way possible, but only in formulating conditions which seem reasonable on physical grounds and which will lead to unique solutions of a type which interest us.

complete set of conditions on φ. We shall obtain all non-vanishing solutions of this problem by constructing them explicitly.

Our method of solving this boundary value problem requires the introduction of the analytic function $f(z)$ of the complex variable $z=x+iy$, the real part of which is the potential function φ:

$$f(z) = \varphi + i\psi.$$

We must reformulate our conditions on φ in terms of $f(z)$. The boundary conditions (4.1) and (4.2) can be written as

$$\begin{aligned}\left(\frac{\partial}{\partial y} - 1\right)\varphi &= \Re e\left(\frac{\partial}{\partial y} - 1\right)(\varphi + i\psi) \\ &= \Re e\left(i\frac{d}{dz} - 1\right)f(z) = \Re e L_1(D)f = 0 \text{ on the positive real axis} \qquad (4.4)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}(\varphi) &= \Re e\frac{\partial}{\partial x}(\varphi + i\psi) \\ &= \Re e\frac{df(z)}{dz} = \Re e L_2(D)f = 0 \text{ on the negative imaginary axis.} \qquad (4.5)\end{aligned}$$

The last step in each case is justified by the fact that $f(z)$ is analytic. The symbol $\Re e$ means, of course, that the real part of what follows is to be taken, and the symbols L_1 and L_2 refer to the linear operators defined by (4.4) and (4.5). ($D=d/dz$.)

The conditions (4.3_∞) at ∞ on φ lead to the following conditions on $f(z)$ at ∞: 1) Because of the fact that φ is uniformly bounded at ∞, the Lemma of Sec. 2 shows that $|df/dz|$ and $|d^2f/dz^2|$ tend to zero along all straight lines in the quarter-plane which are not parallel to its boundaries. 2) Because of the Cauchy-Riemann equations $|df/dz|$ and $|d^2f/dz^2|$ are uniformly bounded in z as $|z|\to\infty$.

We shall make use of the condition (4.3_0) at the origin in the following form: The analytic function $f(z)$ is such that $|df/dz| < M_1/|z|$ and $|d^2f/dz^2| < M_2/|z|^2$ with M_1 and M_2 constants, in a neighborhood of the origin. This statement follows immediately from the conditions (4.3_0) since $r\varphi_x$, $r\varphi_y$, $r^2\varphi_{xx}$, $r^2\varphi_{xy}$ are as a result bounded near the origin and this leads to finite bounds for $|zf'|$ and $|z^2f''|$ through use of the Cauchy-Riemann equations.

Our method of solution depends essentially upon the observation that the linear operators L_1 and L_2 defined by (4.4) and (4.5) have the following property:

$$\Re e L_1L_2(f) = \Re e L_2L_1(f) = 0 \qquad (4.6)$$

on both boundaries of our domain, i.e. on both the positive x-axis and the negative y-axis. This property is an immediate consequence of the linearity and special form of L_1 and L_2 and the boundary conditions (4.4) and (4.5). We are thus led to consider the analytic function $F(z)$ defined by (see (4.4) and (4.5))

$$F(z) = L_2L_1f(z) = \frac{d}{dz}\left(i\frac{d}{dz} - 1\right)f(z). \qquad (4.7)$$

We shall prove the following: *If the analytic function $f(z)$ having the properties we have postulated exists, then $F(z) \equiv Ai/z^2$*, with A an arbitrary real constant; *hence $f(z)$ would satisfy an ordinary differential equation with constant coefficients.*

What are the properties of $F(z)$, assuming that our $f(z)$ exists? From (4.6) we observe that the real part of $F(z)$ vanishes on both boundaries of the quarter-plane (i.e. on the positive real axis and the negative imaginary axis). $F(z)$ can therefore be continued analytically by the reflection process into the entire plane; the result will obviously be a single-valued function whose real part vanishes on the entire real as well as the entire imaginary axis. (Here we make essential use of the fact that the original domain is a sector of angle $\pi/2$.) At the origin $F(z)$ has at most a pole of order two since $|df/dz| < M_1/|z|$ and $|d^2f/dz^2| < M_2/|z|^2$ hold near $z=0$, and these bounds for the derivatives of f in the quarter-plane lead to the bound $|F(z)| < M_3/|z|^2$ in a full neighborhood of the origin as one sees from (4.7) and the fact that $F(z)$ is continued by reflection into a single-valued function in the entire plane. Hence $F(z)$ has at most a pole of order two, and not an essential singularity, at $z=0$. In the same way the conditions at ∞ on $f(z)$ yield for $F(z)$ the condition that $|F(z)|$ is uniformly bounded at ∞. Also $|F(z)|$ tends to zero when $|z| \to \infty$ along any ray which is not parallel to the real or the imaginary axis, since $|df/dz|$ and $|d^2f/dz^2|$ have this property. The only analytic function $F(z)$ with all of these properties is $F(z) = Ai/z^2$, with A an arbitrary real constant (zero included): It is well known that a single-valued analytic function defined in the entire plane is determined by its singularities, which in this case consist of a pole at the origin. This fact, together with the additional conditions on $F(z)$, leads easily to our result.

We can now be certain that the solutions $\varphi(x, y)$ of our potential problem must be the real part of an analytic function $f(z)$ which satisfies the ordinary differential equation

$$\frac{d}{dz}\left(i\frac{d}{dz} - 1\right)f(z) = A\frac{i}{z^2}, \quad A \text{ real.} \tag{4.8}$$

Our problem is therefore reduced to that of determining integration constants in the solution of (4.8) in such a way as to satisfy the boundary conditions (4.4) and (4.5) and the conditions at the origin and at ∞. We shall see that such solutions of (4.8) can be determined, which means that potential functions $\varphi(x, y) = \Re e f(z)$ satisfying our conditions will be shown to exist. It will also be shown that the solutions φ of our problem behave on the water surface at ∞ like $C\cos(x+\alpha)$ in which C and α, the "amplitude" and "phase" of φ, may have any values. Once the phase and amplitude at ∞ are prescribed, however, the solution is uniquely determined.

We proceed to solve the differential equation (4.8) and to fix the constants appropriately. One integration can evidently be carried out at once to yield

$$\left(i\frac{d}{dz} - 1\right)f(z) = -A\frac{i}{z}, \quad A \text{ real.} \tag{4.9}$$

The additive constant which arises through the integration would be imaginary because of the boundary condition (4.4); we have taken it to be zero since it would upon integrating (4.9) give rise only to an additive imaginary constant in the general solution for $f(z)$ and this in turn would contribute nothing to the real part φ of $f(z)$.

A solution $f_1(z)$ of (4.9) for $A=0$ (i.e. a solution of the homogeneous equation) can be found which satisfies all of our conditions. The solution is

$$f_1(z) = Be^{-iz}, \tag{4.10}$$

with B a real but otherwise arbitrary constant, as one can readily verify. The homogeneous differential equation thus furnishes solutions of the problem which are bounded. The corresponding real potential function $\varphi_1 = \Re e f_1(z)$ is, evidently

$$\varphi_1(x, y) = Be^y \cos x. \tag{4.10$'$}$$

We observe that these solutions differ in amplitude but not in phase. They are, in fact, the non-singular solutions of our problem mentioned earlier in this section.

Other types of solutions result from the non-homogeneous equation, i.e. for the case $A \neq 0$ and these will be singular at the origin. One solution of the non-homogeneous equation is given by

$$f(z) = -Ae^{-iz} \int_{+\infty}^{-iz} \frac{e^{-t}}{t} dt, \tag{4.11}$$

in which the path of integration is taken along the positive real axis from $+\infty$, then along a circular arc about the origin, and then along a ray to the point $z\beta_k$ with $\beta_k = -i$, as indicated in Fig. 2. The integral evidently converges. However, it is

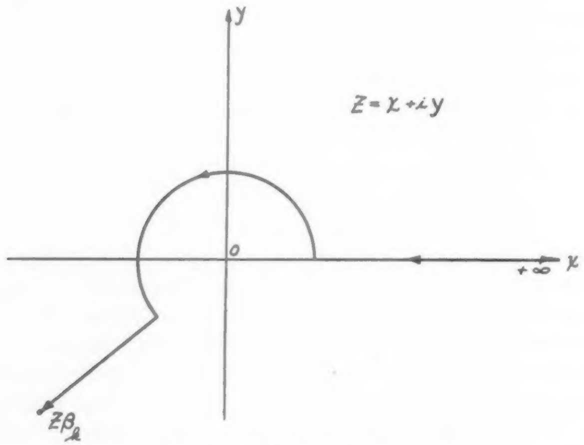

FIG. 2. Path of integration.

necessary to add an appropriately chosen solution of the homogeneous equation in order to satisfy the boundary condition on the negative imaginary axis. (The boundary condition on the real axis is automatically satisfied since $f(z)$ satisfies (4.9) with A real.) From (4.11) we find

$$\frac{df}{dz} = -A\left[-ie^{-iz}\int_{+\infty}^{-iz} \frac{e^{-t}}{t} dt + \frac{1}{z}\right], \tag{4.12}$$

and we are interested in the value of the expression on the right when z is a point on

the negative imaginary axis. One sees readily that the straight line portions of the path of integration (both of which lie on the real axis in this case) contribute purely real quantities to $\int_{\infty}^{-iz}(e^{-t}/t)dt$, but that the semicircular part yields an imaginary contribution of amount[13] plus πi. Thus the real part of the right hand side of (4.12) for z on the negative imaginary axis reduces to the real part of $A\pi e^{-iz}$ there, since $1/z$ and ie^{-iz} are both pure imaginary on this axis and A is real. If, therefore, we add the function $+A\pi i e^{-iz}$ to the right hand side of (4.11) the result

$$f_2(z) = -A\left[e^{-it}\int_{+\infty}^{-iz}\frac{e^{-t}}{t}\,dt - \pi i e^{-it}\right] \tag{4.13}$$

will be a solution of the non-homogeneous equation for which $\Re e\, df_2/dz = 0$ on the negative imaginary axis, i.e. it will be a function satisfying the boundary condition on the imaginary axis.

Thus $f_1(z)$ and $f_2(z)$ as defined in (4.10) and (4.13) are two linearly independent analytic functions which satisfy the boundary conditions. We observe also that $f_1(z)$ and $f_2(z)$ behave in the prescribed manner at the origin: The function $f_1(z)$ is regular there while $f_2(z)$ obviously has a logarithmic singularity.

We have still to check the conditions at ∞. For $f_1(z)$ the conditions are obviously satisfied. To investigate the behavior of $f_2(z)$ as $z\to\infty$ we must consider the asymptotic behavior of the integral $\int_{\infty}^{iz}(e^{-t}/t)dt$. *We shall show in Appendix I that the* integral possesses the following asymptotic representation:[14]

$$\int_{+\infty}^{-iz}\frac{e^{-t}}{t}\,dt \cong 2\pi i - ie^{iz}\left[\frac{1}{z} + \cdots\right]$$

the dots representing higher order terms in $1/z$. Assuming that such a development holds, then $f_2(z)$ clearly possesses the following asymptotic development:

$$f_2(z) \cong -A\pi i e^{-iz} = -A\pi e^{-i(z-\pi/2)} \text{ (with } A \text{ real).} \tag{4.14}$$

The derivatives of $f_2(z)$ have essentially the same asymptotic behavior, as one can readily see. Thus all conditions at ∞ are satisfied. The function $\varphi_2(x, y) = \Re e f_2(z)$ behaves as follows at ∞:

$$\varphi_2(x, y) \cong -A\pi e^{y}\sin x. \tag{4.14'}$$

The solutions $\varphi_1(x, y)$ (cf. (4.10')) and $\varphi_2(x, y)$ are thus out of phase at ∞ and we have therefore achieved one of our main objects. Our conjecture that solutions behaving like φ_2 at ∞ would result by imposing a logarithmic singularity at the origin proves to be correct.

It is important to show that *the set of all analytic functions $f(z)$ which satisfy our boundary and regularity conditions is given by*

$$f(z) = f_1(z) + f_2(z) \tag{4.15}$$

with $f_1(z)$ and $f_2(z)$ the above defined solutions of (4.9) $-f_1$ of the homogeneous equation and f_2 of the non-homogeneous equation. This one proves as follows: Suppose that $g(z)$ is any solution of our problem and set $G(z) = f(z) - g(z)$, with $f(z)$ given by

[13] This is readily seen by expanding e^{-t} in powers of t and observing that the term $1/t$ furnishes the entire value of the integral.

[14] The present case corresponds to the case $\beta_k = -i$ of Appendix II, which in turn arises for $k=2$, $n=2$.

(4.15). It is clear that $G(z)$ satisfies the same boundary conditions as f and g. Since f and g both satisfy (4.9) with the same value of the constant A it follows that $(id/dz-1)G=0$ and the only solution of this equation satisfying our conditions is $G=Ce^{-iz}$, with C real. Thus $g(z)$ could differ from $f(z)$ only by an additive real multiple of $f_1(z)$—that is, g and f are really the same manifold of solutions. We observe that the set of solutions (4.15) contains two real constants A and B which are at our disposal.

From (4.10′) and (4.14′) we conclude that real potential functions solving our problem and having any prescribed amplitude and phase at ∞ can be obtained by superposition of φ_1 and φ_2, since A and B may be chosen arbitrarily, and since our boundary value problem for the function φ is linear and homogeneous. Conversely, it is evident that the constants A and B are uniquely determined when the phase and amplitude of any solution φ at ∞ are prescribed. Once this has been done the complex potential function $f(z)$ is uniquely determined (cf. (4.15) and the remarks concerning it) and hence also the real potential function $\varphi(x, y)$. In other words, our solutions φ exist and are uniquely determined when we prescribe the phase and amplitude at ∞.

We now reintroduce the original space variables by replacing x and y in all relations by mx and my, in which m satisfies the conditions $\sigma^2=gm=2\pi g/\lambda$ (cf. Sec. 3) and σ is the circular frequency. At ∞ our solutions φ_1 and φ_2 have been shown to behave as follows:

$$\varphi_1 = C_1e^{my}\cos mx,$$

$$\varphi_2 \cong C_2e^{my}\sin mx,$$

and consequently the quantity $\lambda=2\pi/m$ is the "wave length" at ∞. This substantiates the remark made in Sec. 3 that the asymptotic relation between the wave length at ∞ and the frequency is the same as that for water which is everywhere (i.e. for all values of x) infinite in depth.

The standing wave solutions φ_1 and φ_2 will be discussed in detail in Sec. 6.

5. Traveling waves over a sloping beach. In this section we shall generalize the method of the preceding section to yield solutions for waves on a beach which slopes at an angle $\pi/2n$ with the horizontal, with n any integer.[15] The method we use is in principle exactly the same as for our previous case of a vertical cliff. The only differences arise from the fact that the differential equation corresponding to (4.11) cannot be obtained quite so easily: it will be, in fact, a differential equation with constant coefficients of order $2n$ instead of one of order two. Naturally, the actual determination of the desired solution will therefore become more and more complicated as n increases, i.e. as the inclination angle of the beach decreases.

We formulate our problem at the outset in terms of the analytic function $f(z)=\varphi+i\psi$. We seek such a function in the sector of angle $\pi/2n$ indicated in Fig. 3. The function $f(z)$ should be regular in the interior of this domain, and have at most a

[15] The problem can be solved for other angles by similar methods, and probably also for any angle by extensions of the theory along the lines of the method of Sommerfeld used in diffraction problems. However, it seems certain that such solutions would be very complicated and would involve functions not easily handled numerically with the tables of functions now available. The cases we discuss in this section are, it happens, amenable to numerical treatment. In any case, for angles less than 90°, it seems certain that the main features of the wave motion will be completely revealed through study of our special cases. For angles greater than 90°—that is, for overhanging cliffs or docks—new features could be expected to arise, and these cases deserve study. As we mentioned in the introduction, H. Lewy [6] has solved the problem for angles $p\pi/2n$ with p any odd integer and n any integer such that $2n>p$.

logarithmic singularity at the origin, which we interpret to mean (cf. the remarks on this point in the preceding section) that $|d^kf(z)/dz^k| < M_k/|z|^k$ for $k=1, 2, \cdots, 2n$, with M_k certain constants. At ∞ we require that $|\Re e f(z)|$ and $|d^kf(z)/dz^k|$ for $k=1, 2, \cdots, 2n$ should remain uniformly bounded when $z\to\infty$ in the sector.[16] As a consequence, all derivatives of $f(z)$ tend to zero along certain rays. On the boundary the conditions on $\varphi(x, y) = \Re e f(z)$ lead, as before, to the following conditions on $f(z)$ at the boundary (cf. (4.4) and (4.5)):

$$\Re e L_1(D)\cdot f(z) = \Re e\left(-\, ie^{-i(\pi/2n)}\frac{d}{dz}\right)f = 0 \quad \text{for} \quad z = re^{-i(\pi/2n)},\ r > 0 \tag{5.1}$$

$$\Re e L_{2n}(D)\cdot f(z) = \Re e\left(i\frac{d}{dz} - 1\right)f = 0 \quad \text{for} \quad z = x > 0. \tag{5.2}$$

By D we mean, of course, differentiation with respect to z. The boundary condition (5.1) should state that the derivative of $\varphi(x, y)$ normal to the bottom vanishes; that it does can be checked easily, for example by inserting $\varphi+i\psi$ for f, replacing d/dz by $\partial/\partial x$, and using the Cauchy-Riemann equations. The condition (5.2) at the free surface is the same as (4.4).

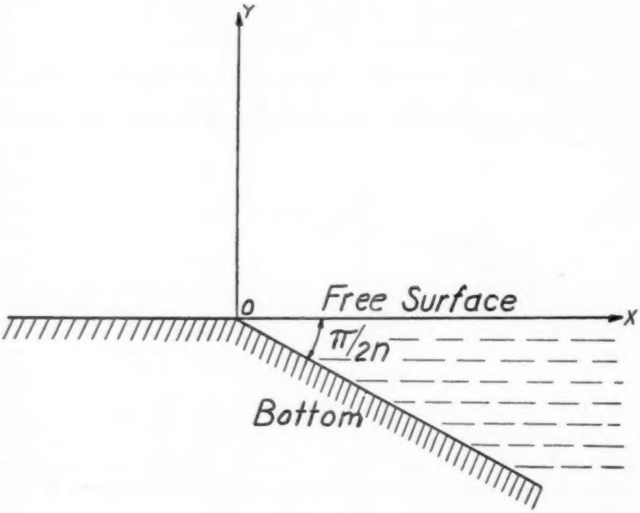

FIG. 3.

In the case treated in the preceding section, the operators L_1 and L_2 had the property $\Re e L_1\cdot L_2 f(z) = \Re e L_2 \cdot L_1 f(z) = 0$ on both boundaries (cf. (4.6)). This is, however, not the case for the corresponding operators L_1 and L_{2n} defined in (5.1) and (5.2). It is necessary for our purposes, in fact, to make use of a set of $2n$ linear operators $L_1, L_2, \cdots, L_{2n-1}, L_{2n}$, of which L_1 and L_{2n} are the first and last members of the sequence, and which are so defined that $\Re e(L_1\cdot L_2\cdot L_3\cdot \cdots L_{2n-1}\cdot L_{2n} f) = 0$ on both boundaries. It would be possible to derive these operators through geometrical constructions and arguments (involving essentially a succession of reflections in the lines $re^{-i(k\pi/2n)}$, $k=1, 2, \cdots$), but we prefer to give them at once and then examine them to see that they have the properties we wish. They are defined as follows:

$$L_k = \begin{cases} (\alpha_k D) & \text{if } k \text{ is odd,} \\ (\alpha_k D - 1) & \text{if } k \text{ is even,} \end{cases} \tag{5.3}$$

[16] It would be possible to weaken this requirement considerably.

in which the α_k are the following complex numbers:

$$\alpha_k = e^{-i\pi[k/n+1)/2}, \qquad k = 1, 2, \cdots, 2n. \tag{5.4}$$

It is useful to bear in mind the location of the points α_k in the complex plane, as given in Fig. 4. These numbers lie on the unit circle spaced at equal angles $\pi/2n$, all of them except $\alpha_{2n}=i$ having negative real parts.

We show that these operators have the required properties. To begin with, we verify at once that the operators L_1 and L_{2n} as given by (5.3) are the same as those given in (5.1) and (5.2). We write:

$$L(D)f = L_1 \cdot L_2 \cdots L_{2n} \cdot f(z) = (\alpha_1 D)(\alpha_2 D - 1) \cdots (\alpha_{2n-1}D)(\alpha_{2n}D - 1)f(z). \tag{5.5}$$

Our object is to show that $\Re e L(D)f=0$ on both boundaries of the sector. We know that $\Re e(\alpha_{2n}D-1)f(z)=0$ on the real axis (condition (5.2)). We proceed to show that the operator $P_1(D)$ defined through (5.5) by $L(D)=P_1(D)\cdot(\alpha_{2n}D-1)$ has all of its coefficients real. It is clear that we may write the polynomial $P_1(D)$ as the product of two factors: $P_1(D)=P_1'(D)P_1''(D)$, with $P_1'(D)$ and $P_1''(D)$ defined as follows:

$$P_1'(D) = [\alpha_1\alpha_{2n-1}D^2][\alpha_3\alpha_{2n-3}D^2] \cdots,$$

$$P_1''(D) = [(\alpha_2 D - 1)(\alpha_{2n-2}D - 1)][(\alpha_4 D - 1)(\alpha_{2n-4}D - 1)] \cdots.$$

That is, we separate the linear factors of P_1 into two groups, one containing all factors for which k is even, the other all those for which k is odd; these two groups are then arranged in the manner indicated. From the definition (5.4) of the α_k (cf. also Fig. 4) it is clear that $\alpha_k = \bar{\alpha}_{2n-k}$, in which the bar over a quantity means that the complex conjugate of the quantity is taken, and also $|\alpha_k| = 1$ for all k. Hence both $\alpha_k \cdot \alpha_{2n-k}$ and $\alpha_k + \alpha_{2n-k}$ are real numbers for all $k = 1, 2, \cdots, 2n-1$, and hence all of the quadratic factors in P_1' and P_1'' obviously have real coefficients. Since $P_1(D)$ contains an odd number of linear factors it follows that either P_1' or P_2' will contain one unpaired linear factor, i.e. the factor containing $\alpha_n D$. But since $\alpha_n = -1$ (cf. (5.4)) it follows that the polynomials P_1' and P_1'' have all their coefficients real and hence also the

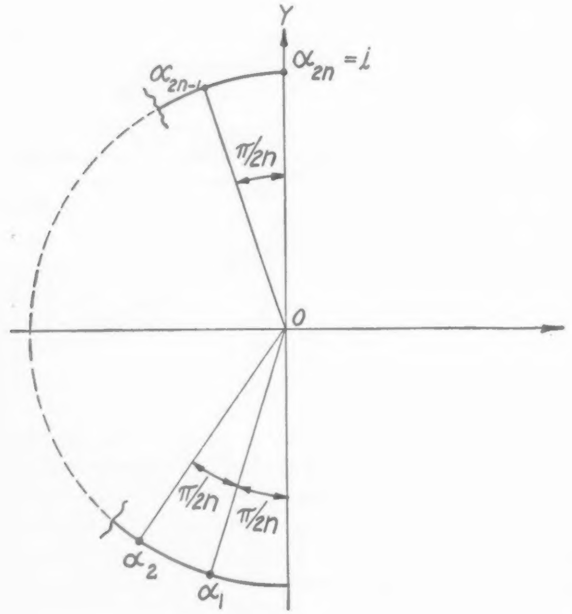

FIG. 4.

polynomial operator $P_1(D)$. Consequently when z varies along the real axis we may write

$$\Re e L(D) f = P_1\left(\frac{\partial}{\partial x}\right) \cdot \Re e(\alpha_{2n} D - 1) f,$$

since D may be replaced by $\partial/\partial x$ in this case and the coefficients of the operator $P_1(\partial/\partial x)$ are all real. Therefore, in view of (5.2) we may write

$$\Re e L(D) f = 0 \quad \text{on the real axis.} \tag{5.6}$$

To prove that the same relation holds along the bottom, i.e. for $z = re^{-i\pi/2n}$, we consider the operator $P_2(D)$ defined by $L(D) = (\alpha_1 D) P_2(D)$. Along $z = re^{-i\pi/2n}$ the operator $D = d/dz$ may evidently be replaced by the operator $e^{i\pi/2n}\partial/\partial r$. By $\partial/\partial r$ is meant, of course, the directional derivative along the bottom line. If, then, D is replaced by $e^{i\pi/2n}\partial/\partial r$ in $P_2(D)$ we observe that $\alpha_2 D, \alpha_3 D, \cdots, \alpha_{2n} D$ become $\alpha_1\partial/\partial r, \alpha_2\partial/\partial r, \cdots, \alpha_{2n-1}\partial/\partial r$ since $\alpha_{k+1}\cdot e^{i\pi/2n} = \alpha_k$ from the definition (5.4) of the α_k. It follows that the operator $P_2(D)$ along $z = re^{i\pi/2n}$ may be replaced by an operator $P_3(\partial/\partial r)$ such that all coefficients of P_3 will be real. This can be seen in exactly the same manner as in the preceding case. We may write, therefore, along this line:

$$\Re e L(D) f(z) = \Re e P_3\left(\frac{\partial}{\partial r}\right)(\alpha_1 D) f(z) = P_3\left(\frac{\partial}{\partial r}\right) \Re e(\alpha_1 D) f(z),$$

and, in view of the boundary condition (5.1), we see that

$$\Re e L(D) f(z) = 0 \quad \text{on} \quad z = re^{-i\pi/2n}. \tag{5.7}$$

Consequently, $\Re e L(D) f(z) = 0$ *on both boundaries.*

We now continue in the same fashion as in the preceding section by introducing the analytic function $F(z)$ defined in our sector by

$$F(z) = L(D) f(z), \tag{5.8}$$

and obtain a differential equation for $f(z)$ by determining $F(z)$ uniquely through use of the properties postulated for $f(z)$. The properties of $F(z)$ are the following: 1) Its real part vanishes on both boundaries of the sector, 2) It is either regular at the origin or it has a pole of order at most $2n$, 3) $|F(z)|$ remains uniformly bounded as $z \to \infty$ in the sector. We know therefore that the derivatives of $F(z)$ will tend to zero along any rays not parallel to the sides of the sector when $|z| \to \infty$, by the Lemma introduced in Sec. 2. The first property has just been established, and the second and third follow immediately from our conditions on $f(z)$ at the origin and at ∞, through the definition (5.8) of $F(z)$.

Because of property 1) and the fact that we are working in a sector of angle $\pi/2n$, with n an integer, it is clear that $F(z)$ can be continued over the boundaries of the sector by successive reflections to yield a single-valued function regular in the entire plane except possibly at the origin where it may have a pole of order $2n$ at most. In addition $|F(z)|$ is bounded at ∞ and the real part of $F(z)$ vanishes on all rays through the origin for which $z = re^{ik\pi/2n}$, $k = 1, 2, \cdots, 4n$. The only analytic function with these properties is $F(z) \equiv A_{2n} i/z^{2n}$, with A_{2n} an arbitrary real constant (zero not excluded). (We rejected a possible additive constant in $F(z)$ through use of the fact

that $F(z)$ tends to zero along certain lines.) It follows therefore that the solutions $f(z)$ of our problem satisfy the differential equation

$$L(D)f = (\alpha_1 D)(\alpha_2 D - 1)(\alpha_3 D)(\alpha_4 D - 1) \cdots (\alpha_{2n-1}D)(\alpha_{2n}D - 1)f = A_{2n} \frac{i}{z^{2n}}, \quad (5.9)$$

with A_{2n} an arbitrary real constant. This differential equation may also be written in the form

$$\alpha_1\alpha_3\alpha_5 \cdots \alpha_{2n-1}D^n(\alpha_2 D - 1)(\alpha_4 D - 1)(\alpha_6 D - 1) \cdots (\alpha_{2n}D - 1)f(z) = A_{2n} \frac{i}{z^{2n}} \cdot (5.10)$$

If we integrate (5.10) once with respect to z on both sides the only change on the left side is that D^n becomes D^{n-1} while the right hand side becomes $-A_{2n}i/(2n-1)z^{2n-1}+C_1$ in which C_1 is an integration constant. But since $|d^kf/dz^k|$ for $k=1, 2, \cdots, 2n$ tends to zero as $|z|\to\infty$ along any ray not parallel to the sides of the sector it follows that $C_1=0$. The same process can be repeated—in particular, the successive integration constants will be zero for the same reason—until we obtain[17]

$$(\alpha_2 D - 1)(\alpha_4 D - 1) \cdots (\alpha_{2n}D - 1)f = A \frac{i}{z^n}, \quad (5.11)$$

with A real (but otherwise arbitrary) since the product $\alpha_1\alpha_3\alpha_5 \cdots \alpha_{2n-1}$ is real. The nth integration would also yield an additional constant on the right hand side of (5.11) which would not necessarily be zero, but which would be pure imaginary. This follows from the boundary condition $\Re e(\alpha_{2n}D-1)f=0$ on the real axis, the fact that $(\alpha_2D-1)(\alpha_4D-1) \cdots (\alpha_{2n-2}D-1)=P_1''(D)$ has only real coefficients (as we have seen a little earlier), and the obvious fact that $\Re e Ai/z^n$ (A real) is zero for real z. However, such an imaginary constant on the right hand side of (5.11) would clearly contribute to the general solution $f(z)$ of (5.11) only an additive pure-imaginary constant which would contribute nothing to the real potential function φ. Consequently we take this constant to be zero.

Once we have the differential equation (5.11) the general procedure as well as the results are exactly analogous to those of the preceding Section 4: the arbitrary constants in the general solution of (5.11) can be fixed so that two types of solutions $f_1(z)$ and $f_2(z)$ are obtained which satisfy the boundary conditions[18] as well as the conditions at the origin and at ∞. The solutions $f_1(z)$ are obtained from the homogeneous equation, while the $f_2(z)$ are solutions of the non-homogeneous equation. The solutions $f_1(z)$ are regular at the origin, while the solutions $f_2(z)$ have a logarithmic singularity there. All solutions $f(z)$ of our problem are then given by $f(z)=f_1(z)+f_2(z)$ and this set of solutions depends only on two real constants which occur as factors in f_1 and f_2. At ∞ the behavior of f_1 and f_2 is such that the real potential functions $\varphi_1=\Re e f_1$ and $\varphi_2=\Re e f_2$ behave like $C_1e^y \cos(x+\alpha_1)$ and $C_2e^y \cos(x+\alpha_2)$ in which C_1 and

[17] The differential equation (5.11) was obtained by H. Lewy in a different way through reflecting in the bottom and then working in a sector of angle π/n instead of one of angle $\pi/2n$. At ∞ Lewy assumes at the outset that the solutions behave like those in water of infinite depth, in contrast to the above treatment in which only the boundedness of certain derivatives is assumed.

[18] That this can be done is far from trivial, since our boundary conditions must be satisfied at all points on the straight lines composing the boundary of the sector.

C_2 are arbitrary, but α_1 and α_2 are fixed and differ by $\pi/2$. It then follows that any solution φ of our problem is uniquely determined once the phase and amplitude of φ at ∞ are prescribed.

The general solution of (5.11) containing n arbitrary constants is of course obtained by straightforward and elementary methods. However, the determination of these constants in order to satisfy the boundary conditions is not entirely trivial, particularly in the case of the solution $f_2(z)$ of the non-homogeneous equation. In Appendix I we discuss the method of determining the integration constants in such a way as to satisfy the boundary conditions; in the present section we simply give the results. The solutions are also seen to satisfy the conditions at the origin and at ∞.

In the homogeneous case, the solution is

$$f_1(z) = \frac{\pi}{(n-1)!\sqrt{n}} \cdot \sum_{k=1}^{n} c_k e^{z\beta_k}, \qquad k = 1, 2, \cdots, n. \tag{5.12}$$

The numerical factor before the summation sign is chosen for convenience in later computations. The constants c_k and β_k are the following complex numbers:

$$\beta_k = e^{i\pi(k/n+1/2)} \tag{5.13}$$

$$\begin{aligned} c_k &= e^{i\pi[(n+1)/4-k/2]} \cot\frac{\pi}{2n} \cot\frac{2\pi}{2n} \cdots \cot\frac{(k-1)\pi}{2n} \quad \text{for} \quad k = 2, 3, \cdots, n, \\ c_1 &= \bar{c}_n. \end{aligned} \tag{5.14}$$

By comparison with (5.4) we note that $\beta_k = \bar{\alpha}_{2k}$. (The bar on $\bar{c}_n$ and $\bar{\alpha}_{2k}$ means that the complex conjugate of these quantities is taken.) The constants c_k are uniquely determined (see Appendix I) within a real multiplying factor.

As $|z| \to \infty$ in the sector, all terms clearly die out exponentially except the term for $k=n$, which is $c_n e^{-iz}$, since all β_k's except β_n have negative real parts. Even the term for $k=n$ dies out exponentially except along lines parallel to the real axis. (The value of c_n, by the way, is $e^{-i\pi(n-1)/4}$ since the cotangents in (5.14) cancel each other for $k=n$.) This term thus yields the asymptotic behavior of $f_1(z)$:

$$f_1(z) \sim \frac{\pi}{(n-1)!\sqrt{n}} \cdot c_n e^{-iz}. \tag{5.15}$$

The general solution $f_2(z)$ of the non-homogeneous equation (5.11) when the real constant A is set equal to one is as follows:

$$f_2(z) = \sum_{k=1}^{n} a_k \left[e^{z\beta_k} \int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\, dt - \pi i e^{z\beta_k} \right]. \tag{5.16}$$

The β_k's are defined by (5.13); and the a_k's are defined by

$$a_k = c_k/(n-1)!\sqrt{n}, \tag{5.17}$$

that is, they are a fixed multiple (for given n) of the c_k's defined by (5.14). The a_k's, like the c_k's, are uniquely determined within a real multiplying factor. The path of integration for all integrals in (5.16) is indicated in Fig. 2. That the points $z\beta_k$ lie in the left half of the complex plane (as indicated) can be seen from our definition (5.13) of the β_k and the fact that z is restricted to the sector $0 \leqq \arg z \leqq \pi/2n$.

The behavior of $f_2(z)$ at ∞ of course depends on the behavior of the integrals in

(5.16). In Appendix II it is shown that these integrals behave asymptotically as follows:

$$\int_{+\infty}^{z\beta_k} \frac{e^{-t}dt}{t} \sim \begin{cases} -\dfrac{e^{-z\beta_k}}{\beta_k}\left(\dfrac{1}{z} + \cdots\right), & \dfrac{\pi}{2} < \arg z\beta_k \leqq \pi, \\ 2\pi i - \dfrac{e^{-z\beta_k}}{\beta_k}\left(\dfrac{1}{z} + \cdots\right), & \pi < \arg z\beta_k \leqq \dfrac{3\pi}{2}\cdot \end{cases} \tag{5.18}$$

Once this fact is established it is clear from (5.17) and (5.16) that $f_2(z)$ behaves asymptotically as follows:

$$f_2(z) \sim \frac{\pi}{(n-1)!\sqrt{n}} \cdot c_n i e^{-iz}, \tag{5.19}$$

since the term for $k = n$ dominates all others (cf. (5.18)) and arg $z\beta_k > \pi$ in this case. Comparison of (5.19) with (5.15) shows that the real parts of $f_1(z)$ and $f_2(z)$ would be 90° out of phase at ∞.

That the derivatives of $f_2(z)$ behave asymptotically in the same fashion as $f_2(z)$ itself is easily seen, since the only terms in the derivatives of a type different from those in (5.16) are of the form b_k/z^k, k an integer $\geqq 1$. Finally, it is clear that $f_2(z)$ has a logarithmic singularity at the origin. Hence $f_1(z)$ and $f_2(z)$ satisfy all requirements. Just as in the 90° case (cf. the preceding section) it can now be readily seen that $f(z) = b_1 f_1(z) + b_2 f_2(z)$, with b_1 and b_2 any real constants, contains *all* standing wave solutions of our problem. The proof that the real potential function $\varphi = \Re e f(z)$ is uniquely determined once the phase and amplitude at ∞ are fixed can also be carried out exactly as in the previous section for the 90° case.

The relations (5.15) and (5.19) yield for the asymptotic behavior of the real potential functions φ_1 and φ_2 the relations:

$$\varphi_1(x, y) = \Re e f_1 \sim \frac{\pi}{(n-1)!\sqrt{n}} e^y \cos\left(x + \frac{n-1}{4}\pi\right) \tag{5.20}$$

$$\varphi_2(x, y) = \Re e f_2 \sim \frac{\pi}{(n-1)!\sqrt{n}} e^y \sin\left(x + \frac{n-1}{4}\pi\right) \tag{5.21}$$

when it is observed that $c_n = e^{-i\pi(n-1)/4}$. It is now possible to construct either standing wave or progressing wave solutions which behave at ∞ like the known solutions for steady progressing waves in water which is everywhere infinite in depth. In particular we observe that it makes sense to speak of the wave length at ∞ in our cases and that the relation between wave length and frequency satisfies asymptotically the relation which holds everywhere in water of infinite depth. For this, it is only necessary to reintroduce the original space variables by replacing x and y by mx and my, with $m = \sigma^2/g$ (cf. Sec. 3), and to take note of (5.20) and (5.21).

Finally, we write down a solution $\Phi(x, y; t)$ which behaves at ∞ like $e^y \cos(x + t + \alpha)$, i.e. like a steady progressing wave moving toward shore:

$$\Phi(x, y; t) = A\left[\varphi_1(x, y)\cos(t + \alpha) - \varphi_2(x, y)\sin(t + \alpha)\right]. \tag{5.22}$$

The solutions (5.22) will be discussed numerically in Sec. 8 for the case of a beach sloping at an angle of 6° (i.e. for the case $n = 15$).

6. Numerical discussion of standing wave solutions for 90°, 45°, and 6° slopes. In this section we give graphs of the two types of standing wave solutions for the case of a vertical cliff (cf. Sec. 4) and for bottom slopes of 45° and 6°. We continue to make use of the dimensionless variables of Sec. 3. In particular, it should be recalled that the variable x means the distance from shore divided by $\lambda/2\pi$, in which λ is the wave length at infinity. In other words the quantity x is proportional to the wave length at ∞.

In the case of the vertical cliff, or 90° case, two standing wave solutions are given by

$$\Phi_1(x, y; t) = \pi e^{it} e^y \cos x \tag{6.1}$$

and

$$\Phi_2(x, y; t) = e^{it} e^y \left[\cos x \int_\infty^x \frac{\cos \xi}{\xi} d\xi + \sin x \int_\infty^x \frac{\sin \xi}{\xi} d\xi + \pi \sin x \right], \tag{6.2}$$

As one can readily verify, either from (4.10) in Sec. 4 or from (5.12) in Sec. 5 with $n=1$.

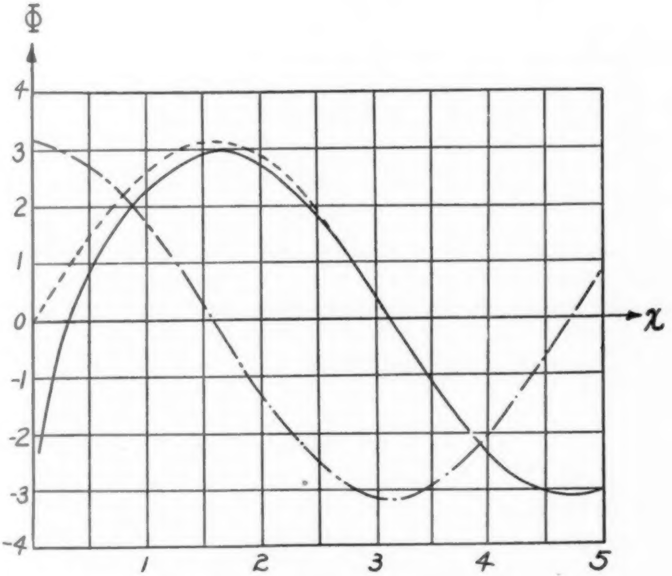

FIG. 5. Standing waves for a vertical cliff.

$\Phi_1(x, 0, 0)$ — - — - $\quad x$ = distance from shore/$(\lambda/2\pi)$

$\Phi_2(x, 0, 0)$ ——— $\quad \lambda$ = wavelength at ∞

$\pi \sin x$ - - - - -

The functions Φ_1 and Φ_2 for $y=0$ and $t=0$ are plotted[19] in Fig. 5 together with the function $\pi \sin x$, which yields the asymptotic behavior of Φ_2 on the surface. The most

[19] The curves for Φ_1 and Φ_2 differ from the corresponding surface elevations η_1 and η_2 (cf. (1.4)) only by a phase shift and a constant multiplier.

notable feature of the curves of Fig. 5 is that the standing wave solution Φ_2, although it tends to become negative infinite as $x \to 0$, nevertheless differs little from the function $\pi \sin x$ until a point within a distance from the cliff of a fraction of the wave length at ∞ is reached. In other words, the asymptotic development of Φ_2 for x large holds with good accuracy until x is rather small. In addition, the last maximum of Φ_2 (i.e. the crest nearest to the cliff) has an amplitude slightly *less* by a little more than 10 per cent than the amplitude at ∞. The distance between the two zeros of Φ_2 nearest to shore is also 10 per cent less than half the wave length at ∞.

An interesting additional fact which is not difficult to prove is that the velocity of the water does not decrease exponentially with the depth y (as it does when no cliff is present), but only like $1/y$. The presence of the cliff thus reinforces the velocity. (The velocity of the water along the cliff is given by $\Re e(idf(z)/dz)$ with $f(z)$ defined by (4.15); the result indicated follows by calculating idf/dz and using the known asymptotic behavior of the complex integral which occurs.)

The standing wave solution Φ_1, Φ_2 for the case of a beach sloping at 45° are obtained from (5.12) and (5.16) with $n=2$. For Φ_1 we write

$$\Phi_1(x, y; t) = \frac{\pi}{\sqrt{2}} e^{it} \Re e\left[e^{i\pi/4} e^{-z} + e^{-i\pi/4} e^{-iz}\right]. \tag{6.3}$$

The unbounded standing wave solution is given by

$$\Phi_2(x, y; t) = \frac{e^{it}}{\sqrt{2}} \Re e\left\{e^{i\pi/4}\left[e^{-z}\int_\infty^z \frac{e^{-t}}{t}\,dt - \pi i e^{-z}\right]\right.$$
$$\left. + e^{i\pi/4}\left[e^{-iz}\int_\infty^{-iz} \frac{e^{-t}}{t}\,dt - \pi i e^{-iz}\right]\right\}. \tag{6.4}$$

The surface values of Φ_1 and Φ_2 for $t=0$ are plotted in Fig. 6. These curves are obtained by using tables of the functions C_i, S_i, and E_i[20] computed by the Mathematical Tables Project [7]. In fact, Φ_2 can be written in the form

$$\Phi_2(x, y; t) = \frac{e^{it}e^{y}}{\sqrt{2}}\left\{C_i(x)\left[\sin x - \cos x\right]\right.$$
$$\left. - \left[\frac{\pi}{2} + S_i(x)\right]\left[\cos x + \sin x\right] - e^{-x}E_i(x)\right\}. \tag{6.4'}$$

At ∞, $\Phi_1(x, 0; 0)$ behaves like $(\pi/\sqrt{2})(\cos x - \sin x)$ while $\Phi_2(x, 0; 0)$ behaves like $-(\pi/\sqrt{2})(\cos x + \sin x)$.

The same general comments can be made about the curves of Fig. 6 as were made for those of Fig. 5. In particular, the minimum of Φ_2 at $x=1$ is about 10 per cent less in numerical value than the amplitude of Φ_2 at ∞, but Φ_2 differs very little from its asymptotic representation until x is quite small. The regular solution Φ_1 attains a maximum on shore which is $\sqrt{2}$ times the amplitude at ∞. The distance between successive zeros shortens on approaching shore, as in the preceding case, but the shortening is now more pronounced.

[20] We use i unconventionally as a subscript to avoid confusion with $i=\sqrt{-1}$; in M.T.P. [7] the notation Ci, Si, and Ei is used.

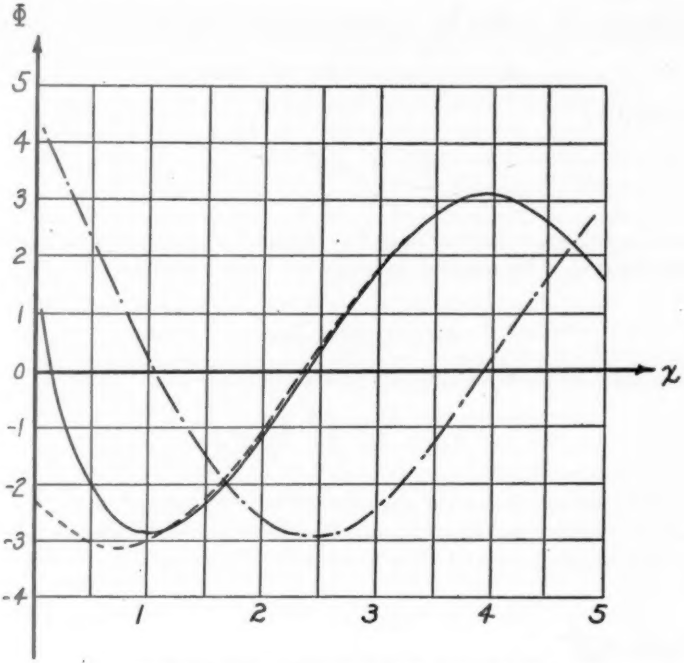

FIG. 6. Standing waves for a 45° bottom slope.

$\Phi_1(x, 0, 0)$ — · — · x = distance from shore/($\lambda/2\pi$)
$\Phi_2(x, 0, 0)$ ——— λ = wavelength at ∞
$-\pi/\sqrt{2}(\cos x + \sin x)$ - - - -

Finally, we describe the two types of standing wave solutions for the case of a bottom slope of 6°. These solutions are obtained from (5.12) and (5.16) by taking $n=15$. The regular standing wave Φ_1 is taken in the form (cf. (5.12)):

$$\Phi_1(x, y; t) = e^{it}\Re e f_1(z) \tag{6.5}$$

with $f_1(z)$ defined by

$$\begin{aligned} f_1(z) &= \sum_{k=1}^{15} c_k e^{z\beta_k}, \\ \beta_k &= e^{i\pi(k/15+1/2)}, \\ c_k &= e^{-i\pi k/2} \cot\frac{\pi}{30} \cdots \cot\frac{(k-1)\pi}{30}, \quad k \neq 1, \\ c_1 &= -i, \end{aligned} \tag{6.6}$$

The fifteen quantities c_k are alternately real and pure imaginary and they vary in absolute value from 1 to approximately 819. For large values of x the function $\Phi_1(x, 0; 0)$ behaves as follows (cf. (5.20)):

$$\Phi_1(x, 0; 0) \sim \pi \sin x. \tag{6.7}$$

The standing wave solution Φ_2, which is infinite at the shore line, is defined by

$$\Phi_2(x, y; t) = -14!\sqrt{15}\, e^{it}\Re e f_2(z) \tag{6.8}$$

with $f_2(z)$ defined by

$$f_2(z) = \frac{1}{14!\sqrt{15}} \sum_{k=1}^{15} c_k \left\{ e^{z\beta_k} \int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\, dt - \pi i e^{z\beta_k} \right\} \tag{6.9}$$

in which the c_k and β_k are defined as in (6.6). The path of integration is shown in Fig. 2 of a preceding section. This solution behaves for large x and for $y=0$, $t=0$ as follows (cf. (5.21)):

$$\Phi_2(x, 0; 0) \sim \pi \cos x. \tag{6.10}$$

In order to determine Φ_2 numerically the function

$$E(z) = e^z \int_z^\infty \frac{e^{-t}}{t}\, dt$$

was computed for values of z on the rays $z = re^{i\pi(k/15+1/2)}$, $k=1, 2, \cdots, 7$.

The function $E(z)$ defined by (6.11) has been tabulated for values of z in the second quadrant by the Mathematical Tables Project [8]. However, the entries in this table

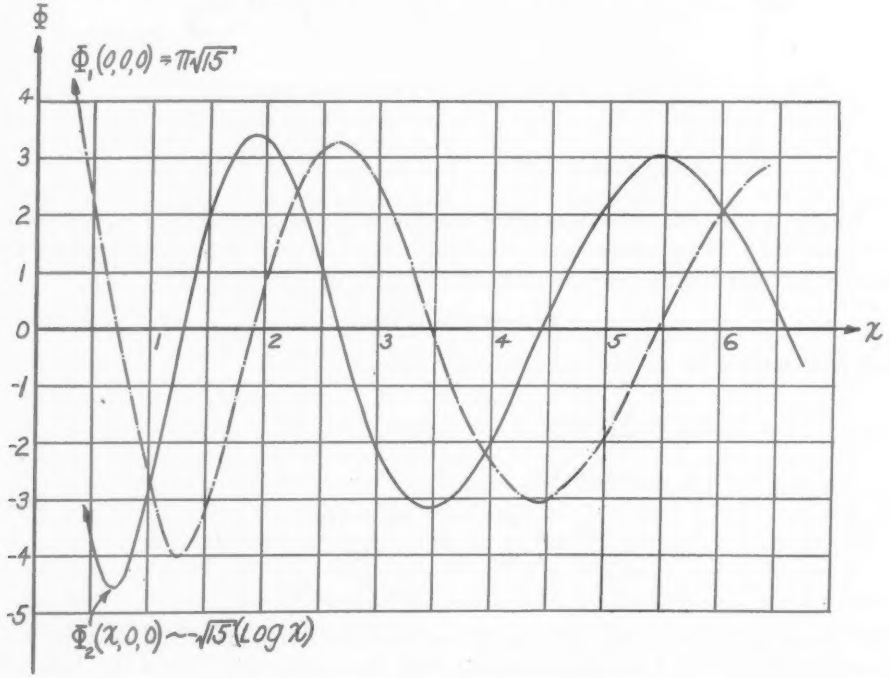

FIG. 7. Standing waves for a 6° bottom slope.

$\Phi_1(x, 0, 0)$ — - — - x = distance from shore/$(\lambda/2\pi)$

$\Phi_2(x, 0, 0)$ ——— λ = wavelength at ∞

are for a rather wide rectangular net of points in the z-plane. We therefore found it necessary to compute $E(z)$ along the above described rays as follows: For $r=|z|=3.00$ and $r=3.50$, we interpolated in the M.T.P. tables of $E(z)$ by means of a Taylor series expansion about the nearest tabulated points. (The derivatives of $E(z)$ are easily calculated.) The values at $r=3.00$ and 3.50 were checked by using them to compute the values at $r=3.25$. Power series developments were thus obtained for points along each of the seven rays for the range $1.00 \leqq r \leqq 7.00$ at an interval of 0.25, which means that more than 100 power series were derived. A table was then constructed by using these series to obtain values at intermediate points. Since integrals of this form would always occur in solving differential equations having constant coefficients and rational right hand sides, it seems worth while to include this table as an Appendix. The table is believed to be accurate within one unit in the fourth decimal place. The table would, of course, also be useful if computations were to be made for beach slopes at angles other than 6°.

Fig. 7 shows the surface values of Φ_1 and Φ_2 near shore, while Figs. 8 and 9 are graphs of Φ_1 and Φ_2 together with their asymptotic representation for large x. Again the same general comments are in order as in the preceding cases, except that it is now necessary to go further out from shore to obtain close agreement with the asymptotic solution in deep water. This is not very surprising since the depth of the

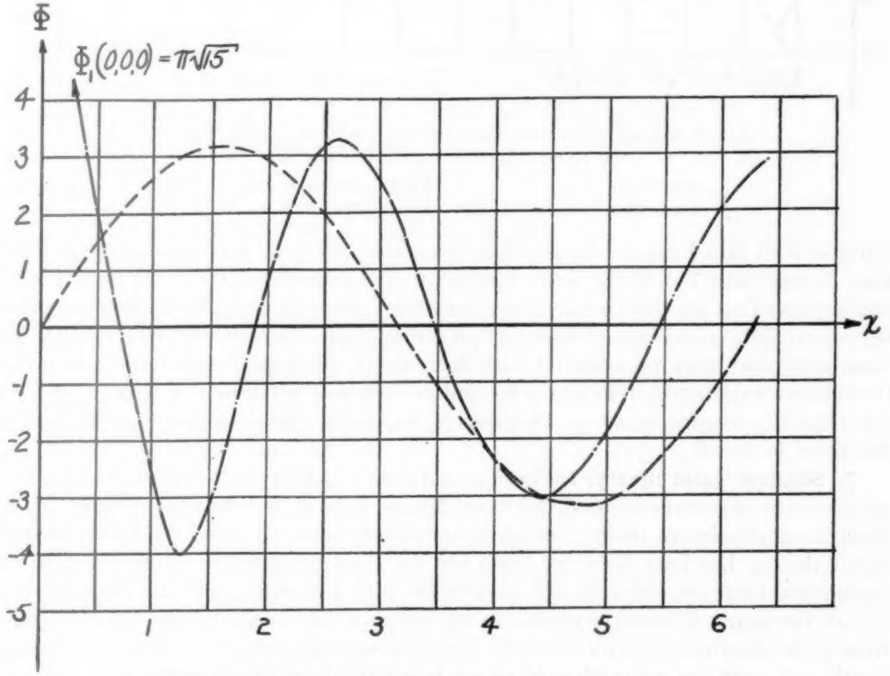

FIG. 8. Comparison of standing wave solution for a 6° slope, Φ_1, with its asymptotic limit $\pi \sin x$ (see Fig. 7).

$\Phi_1(x, 0, 0)$ —·—· x = distance from shore/$(\lambda/2\pi)$

$\pi \sin(x)$ ---- λ = wavelength at ∞

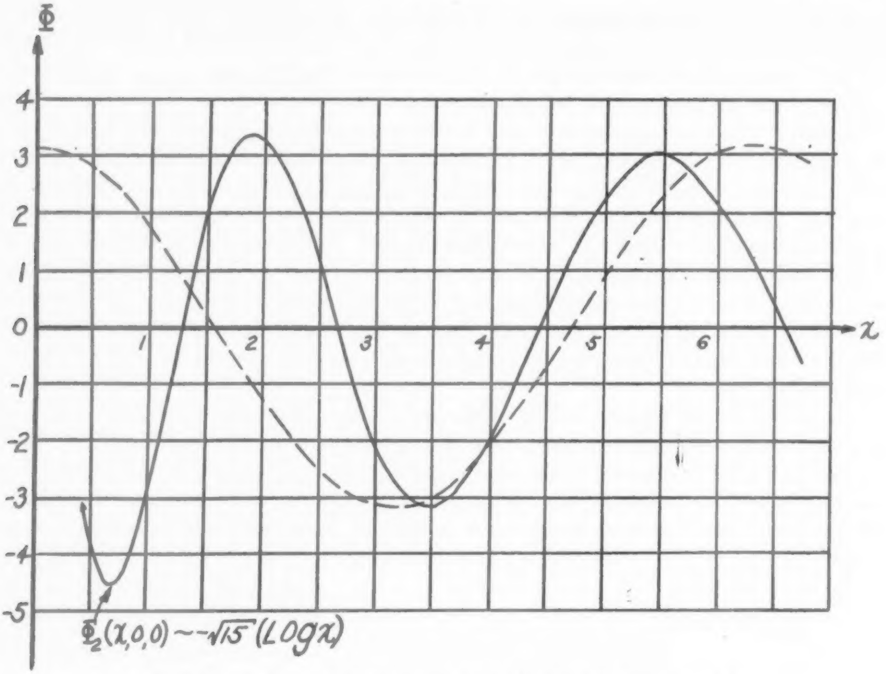

FIG. 9. Comparison of standing wave solution for a 6° slope, Φ_2, with its asymptotic limit $\pi \cos x$ (see Fig. 7).

$\Phi_2(x, 0, 0)$ ——— x = distance from shore/$(\lambda/2\pi)$

$\pi \cos (x)$ - - - - λ = wavelength at ∞

water at $x=7$ (the largest x for which we have numerical values) because of the small slope is less than 1/8 of the wave length at ∞. Comparison of Fig. 7 with Figs. 5 and 6 shows that the distance between successive zeros near shore has now shortened very materially as compared with the preceding cases. In fact, this effect would become more and more pronounced with decrease in the slope.[21] We observe also that the relative maximum of Φ_2 at $x=5.4$ and the relative maximum of Φ_1 at $x=6.5$ are less than the amplitude at ∞. Progressing waves for the case of a 6° slope will be discussed in Sec. 8.

7. Shallow water theory. In some gravity wave problems it is possible to obtain an accurate approximation to the exact linear theory by relatively simple means. Such an approximate theory, which is commonly referred to as the linear shallow water theory, has long been the basis for the theoretical study of the tides in the oceans and large estuaries. In this section we give a derivation of the shallow water theory for water of variable depth; in the next section we shall compare the results from both theories numerically in the case of progressing waves in water of constant depth and over a beach with a 6° slope. In what follows we consider only the two-dimensional case. We revert also to the original space and time variables.

[21] See the next section, where an approximate theory valid for water in which the depth is small compared with the wave length is discussed.

The exact linear theory requires the determination of a potential function $\Phi(x, y; t)$ in the region indicated in Fig. 10). As before, $y=0$ is the original undisturbed surface of the water and $y=-h(x)$ is the bottom profile. The elevation of the

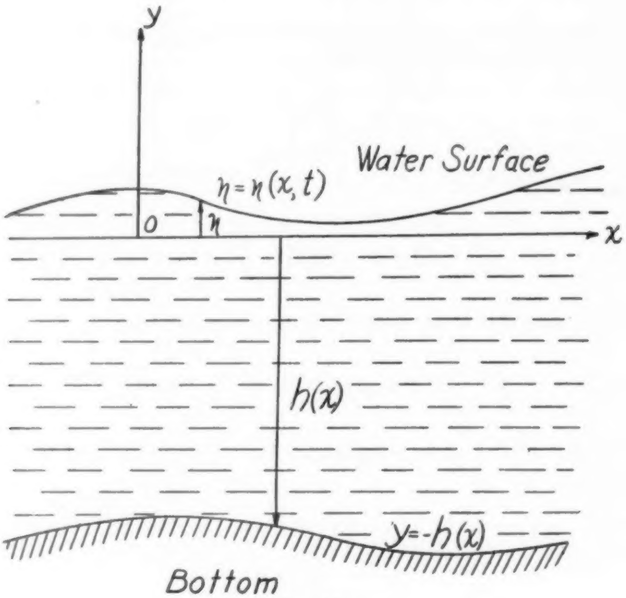

FIG. 10.

free surface in the course of the motion is denoted by $\eta(x, t)$. The conditions to be satisfied by Φ are (cf. Sec. 1).

$$\Phi_{xx} + \Phi_{yy} = 0 \qquad \text{for} \quad 0 \geqq y \geqq -h(x) \tag{7.1}$$

$$\Phi_{tt} = -g\Phi_y \qquad \text{for} \quad y = 0 \tag{7.2}$$

$$\Phi_y = -h_x\Phi_x \qquad \text{for} \quad y = -h(x). \tag{7.3}$$

For the purposes of the present section it is not necessary to formulate conditions at ∞. The surface elevation η is given by

$$\eta(x, t) = \frac{1}{g}\cdot \Phi_t \Big|_{y=0}. \tag{7.4}$$

In the following derivation it is convenient to denote quantities evaluated for $y=0$ by a bar over the quantity, and for $y=-h(x)$ (i.e. at the bottom) by a bar under the quantity. Thus conditions (7.2) and (7.3) could be written in the form $\overline{\Phi}_{tt}=-g\overline{\Phi}_y$ and $\underline{\Phi}_y=-h_x\underline{\Phi}_x$.

We may write

$$\int_{-h(x)}^{0} \Phi_{yy}dy = \overline{\Phi}_y - \underline{\Phi}_y = \overline{\Phi}_y + h_x\underline{\Phi}_x \tag{7.5}$$

from (7.3). On the other hand we may also write

$$\int_{-h(x)}^{0} \Phi_{yy} dy = -\int_{-h(x)}^{0} \Phi_{xx} dy = -\frac{\partial}{\partial x}\int_{-h(x)}^{0} \Phi_x dy + h_x \underline{\Phi}_x \tag{7.6}$$

by using (7.1) and observing that the lower limit $h(x)$ depends on x. By equating the right hand sides of (7.5) and (7.6) we find

$$\bar{\Phi}_y = -\frac{\partial}{\partial x}\int_{-h(x)}^{0} \Phi_x dy. \tag{7.7}$$

We consider next the relation

$$\int_{-h(x)}^{0} \Phi_x dy = h\underline{\Phi}_x - \int_{-h}^{0} y\Phi_{xy} dy \tag{7.8}$$

obtained through integration by parts, and also the relation

$$\int_{-h(x)}^{0} h(x)\Phi_{xy} dy = h(x)\bar{\Phi}_x - h(x)\underline{\Phi}_x. \tag{7.9}$$

The quantity $\underline{\Phi}_x$ in (7.8) can be eliminated by using (7.9), and this in turn lead-through use of (7.7) to

$$\begin{aligned}\bar{\Phi}_y &= -\frac{\partial}{\partial x}\left[h(x)\bar{\Phi}_x - \int_{-h(x)}^{0} (y + h)\Phi_{xy} dy\right] \\ &= -(h\bar{\Phi}_x)_x + \int_{-h}^{0} [(y + h)\Phi_{xxy} + h_x \Phi_{xy}] dy.\end{aligned}$$

From the boundary condition (7.2) we may replace $\bar{\Phi}_y$ by $-1/g\bar{\Phi}_{tt}$ to obtain

$$\bar{\Phi}_{tt} = (gh\bar{\Phi}_x)_x - g\int_{-h}^{0} [(y + h)\Phi_{xxy} + h_x\Phi_{xy}] dy. \tag{7.10}$$

Up to now we have made no assumptions in addition to those made in deriving the exact linear theory—we have simply carried out integrations with respect to y in such a way as to yield (7.10). We now assume that Φ_{xxy} *and* Φ_{xy} *are bounded for all* x *and* t, *and for* $-h \leqq y \leqq 0$, *and that* h_x *is small of the same order of magnitude as* h. It follows at once that (7.10) may be written in the form

$$\bar{\Phi}_{tt} = (gh\bar{\Phi}_x)_x + O(h^2) \tag{7.11}$$

in which $O(h^2)$ is a quantity of order h^2. Thus, if the depth h and slope h_x are sufficiently small, the surface value $\bar{\Phi}$ of the potential function $\Phi(x, y; t)$ should be given with good approximation by the differential equation

$$\bar{\Phi}_{tt} = (gh\bar{\Phi}_x)_x. \tag{7.12}$$

In the case h=const. (7.12) is the wave equation in one space variable with $c = (gh)^{1/2}$ as propagation speed. Equation (7.12) is the differential equation of the

linear shallow water theory.[22] The above derivation of the theory is due to F. John.

Instead of $\bar{\Phi}$ we may introduce the surface elevation $\eta(x, t) = \bar{\Phi}_t/g$ (cf. (7.4)) as dependent variable; this leads easily to the equation

$$\eta_{tt} = (gh\eta_x)_x. \tag{7.13}$$

It is possible to derive the shallow water theory in such a way as to obtain in place of (7.10) a relation in which the integral on the right hand side is replaced by a power series in h which converges if the initial conditions (i.e. conditions for $t=0$) are assumed to be sufficiently smooth. That is, it would not be necessary in this version of the theory to assume at the outset that Φ_{xxy} and Φ_{xy} are bounded for all time.

We proceed to derive standing wave solutions of (7.12) for the case in which

$$h(x) = qx, \qquad q = \text{const.}, \tag{7.14}$$

i.e. for the case of a uniformly sloping bottom profile.[23] For this purpose it is convenient to set

$$\bar{\Phi}(x, t) = e^{i(\sigma t+\epsilon)}Z(x). \tag{7.15}$$

The function $Z(x)$ satisfies

$$(xZ_x)_x + \frac{m}{g}Z = 0, \qquad m = \frac{\sigma^2}{g}, \tag{7.16}$$

in which the quantity m has the same definition in terms of σ as in the exact theory of the preceding sections (cf. Sec. 3).

The differential equation (7.16) is a Bessel equation with the general solution

$$Z(x) = AJ_0\left(2\sqrt{\frac{mx}{q}}\right) + BY_0\left(2\sqrt{\frac{mx}{q}}\right). \tag{7.17}$$

J_0 and Y_0 are the regular and the singular Bessel functions of order zero respectively; thus Y_0 has a logarithmic singularity at $x=0$. For large values of x these functions are well known to behave as follows:

$$J_0\left(2\sqrt{\frac{mx}{q}}\right) \sim \frac{\cos\left(2\sqrt{\frac{mx}{q}} - \frac{\pi}{4}\right)}{\sqrt{\pi\sqrt{\frac{mx}{q}}}}$$

---

[22] The derivation often given for this theory (cf. for example [4] p. 254) starts with the assumption that the pressure p is determined by the same relation as in hydrostatics, i.e. $p=g(\eta-y)$. One would be inclined to think that such a relation would be on the whole more accurate the deeper the water since all motions die out in the depth, but it is easily seen on the basis of the simplest examples that the approximate theory cannot be accurate in sufficiently deep water: The exact solutions for steady periodic waves in water of constant depth yield waves in which the wave length depends essentially on the frequency, but the steady waves given by (7.12) in the case h=const. are without dispersion. In other words, the derivation of the approximate theory by means of the hydrostatic assumption is open to criticism since it does not indicate clearly the essential role played by the depth in determining the accuracy of the approximation. Cisotti has given another derivation of the shallow water theory (cf. [10] pp. 357, 379) which does not start with the hydrostatic assumption.

[23] This theory is, of course, well known. See, for example [4], p. 291.

$$Y_0\left(2\sqrt{\frac{mx}{q}}\right) \sim \frac{\sin\left(2\sqrt{\frac{mx}{q}} - \frac{\pi}{4}\right)}{\sqrt{\pi\sqrt{\frac{mx}{q}}}}. \tag{7.18}$$

With this theory it is therefore not possible to obtain either standing waves or progressing waves with non-vanishing amplitudes at ∞, in sharp contrast to solutions given by the exact theory. Also, for large values of x the wave length (defined as the distance between successive nodes, say) as given by the approximate theory would be roughly $2\pi(gx/m)^{1/2}$ and would therefore increase indefinitely with x. This is also in marked contrast with our exact solutions, in which the wave length at ∞ tends to a constant.

Nevertheless, it is possible to reintroduce the time factor and obtain for $\eta(x, t)$ solutions which have the form of progressing waves traveling toward shore, as follows:

$$\bar{\Phi}(x, t) = A\left[\cos(\sigma t - \epsilon)Y_0(2\sqrt{mx/q}) + \sin(\sigma t - \epsilon)J_0(2\sqrt{mx/q})\right]. \tag{7.19}$$

One could expect that such a solution might furnish in some cases at least a fair approximation to the exact solution for a not too large range of values of x. In particular, we note that the singularity at the origin (i.e. on the shore line) is of the same type as in the exact theory. In the next section a numerical comparison with the exact theory will be made.

8. Comparison of exact and shallow water theories. In the present section we compare the results obtained using the shallow water theory derived in the preceding section with those of the exact theory. We consider progressing wave solutions first for the case of water of uniform but finite depth and then for the case of a bottom slope of 6°. In this section it is preferable to use the original independent variables rather than the dimensionless variables of Sec. 3.

In the case of water of uniform depth h the velocity potential $\Phi(x, y; t)$ is given by (cf. [4], pp. 363–368)

$$\Phi(x, y; t) = A\,\frac{\cosh m(y + h)}{\cosh mh}\,\sin(mx + \sigma t), \tag{8.1}$$

in which h represents the constant depth of the water. The undisturbed surface of the water is the line $y=0$. This potential function evidently satisfies the condition $\partial\Phi/\partial y=0$ at the bottom $y=-h$. The free surface condition (1.2) is also satisfied if

$$\sigma^2 = mg \tanh mh, \tag{8.2}$$

or as we may also write

$$\sigma^2/gh = m^2\left(1 - \frac{m^2h^2}{3} + \cdots\right). \tag{8.3}$$

If we introduce the phase velocity $c=\sigma/m$ we may write

$$c = \sqrt{gh\left(1 - \frac{m^2h^2}{3} + \cdots\right)} \tag{8.4}$$

or

$$c = \sqrt{gh\left[1 - \frac{1}{3}\left(\frac{2\pi h}{\lambda}\right)^2 + \cdots\right]} \tag{8.5}$$

if the wave length $\lambda = 2\pi/m$ is used. The relations (8.2) to (8.5) characterize the type of dispersion encountered in this case.[24] If the ratio of wave length λ to depth h is large, the relation (8.5) for the phase velocity c becomes

$$c \cong \sqrt{gh}. \tag{8.6}$$

In fact, if $\lambda/h = 5$, $c = .82(gh)^{1/2}$; while if $\lambda/h = 10$, $c = .94(gh)^{1/2}$.

In the same case of constant depth the shallow water theory gives the following approximation $\bar{\Phi}(x, t)$ to the surface value $\Phi(x, 0; t)$:

$$\bar{\Phi} = A \sin(mx + \sigma t). \tag{8.7}$$

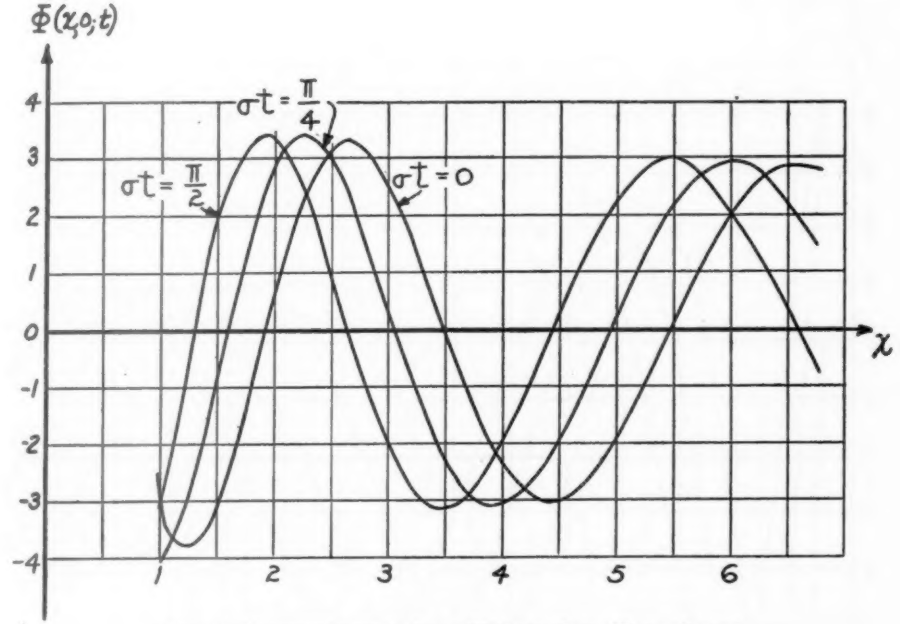

FIG. 11. Progressing waves for a 6° bottom slope. (Exact theory.)
λ = wavelength at ∞
x = distance from shore/$(\lambda/2\pi)$

This is a solution of (7.12) only if

$$\sigma^2/gh = m^2, \tag{8.8}$$

i.e. if the phase velocity $c = \sigma/m$ is given by

$$c = \sqrt{gh}. \tag{8.9}$$

[24] This theory appears to be due to Airy [1], who published it in 1845.

In other words, the approximate solution (8.7) furnished by the shallow water theory yields the phase velocity with good accuracy only if the ratio of the wave length to the depth is sufficiently great, as we see by comparing (8.9) with (8.5). In fact, (8.9) yields c correct, within about 5 per cent, only if the wave length is more than ten times the depth. For steady waves, therefore, the approximate theory is accurate only if the water is shallow compared with the wave length. For this reason the approximate theory is sometimes referred to as the long wave-shallow water theory.

We turn next to a comparison of the results of the two theories for progressing waves over a beach sloping at 6°. The progressing wave solution of the exact theory is given by (5.22), which behaves at ∞ like $\pi \sin (mx+\sigma t)$. Graphs of the numerical solutions for times $\sigma t=0$, $\pi/4$, and $\pi/2$ are shown in Fig. 11. (Again we note that the dimensionless variables of Section 3 are *not* used.) We observe that the "amplitude" of the progressing wave increases as the wave moves from the point $2\pi x/\lambda=6$

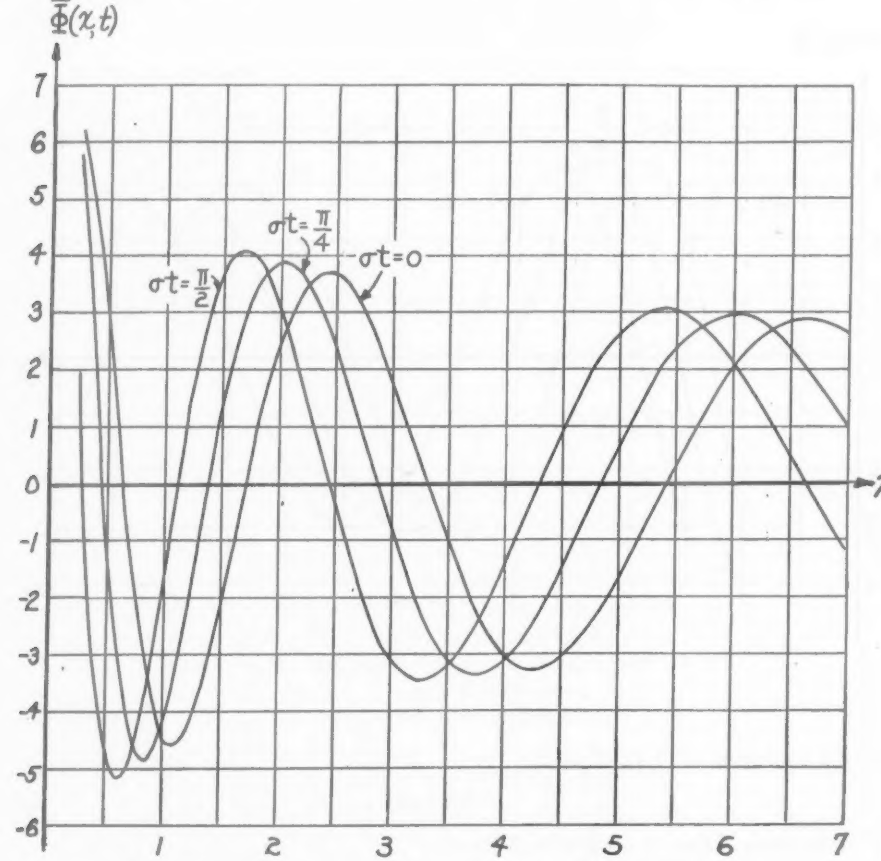

FIG. 12. Progressive waves for a 6° bottom slope (shallow water theory).
λ = wavelength at ∞ in exact linear case
x = distance from shore/$(\lambda/2\pi)$

toward shore. However, the maximum at this point is 6 per cent less[25] than the amplitude at ∞.

In the preceding section it was already explained that the shallow water theory could not, in principle, furnish a good approximation to the progressing waves in the present case over the whole range of values of x since the amplitude tends to zero at ∞ in this theory. We have therefore chosen to make a comparison by assuming that the circular frequency σ is the same for both theories and that both solutions have the same relative maximum for $\sigma t = \pi/4$ and $2\pi x/\lambda = 6$. This yields for the approximate solution $\bar{\Phi}(x, t)$ the following (cf. (7.19)):

$$\bar{\Phi}(x, t) = -14.5\left[\cos(\sigma t - .98)Y_0\left(2\sqrt{\frac{mx}{q}}\right) + \sin(\sigma t - .98)J_0\left(2\sqrt{\frac{mx}{q}}\right)\right]. \tag{8.10}$$

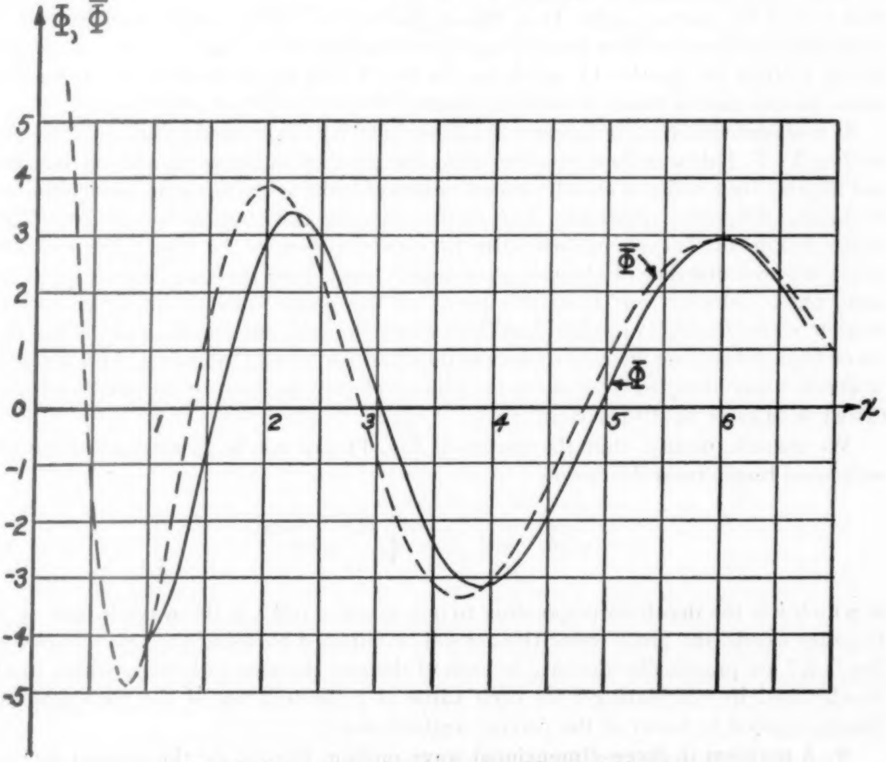

FIG. 13. Comparison of surface values from both theories.

Φ = Exact linear theory ———— λ = wavelength at ∞ (exact theory)
$\bar{\Phi}$ = Shallow water theory — — — — x = distance from shore/$(\lambda/2\pi)$
Phase and amplitude are the same at $(x = 6, \sigma t = \pi/4)$
Frequencies are equal

[25] The author is told that this effect has been observed experimentally.

Fig. 12 is a graph of $\bar{\Phi}$ for $\sigma t=0$, $\pi/4$, $\pi/2$. The curves have the same general appearance as those of Fig. 11, but the amplitude increases somewhat more rapidly in the present case. This is brought out more directly by Fig. 13, which gives graphs of Φ and $\bar{\Phi}$ for $\sigma t=\pi/4$. The two theories agree fairly well as the shore is approached, although the amplitude given by the approximate theory at $2\pi x/\lambda \sim 2$ is about 15 per cent greater than that furnished by the exact theory. We have at our disposal the information necessary to describe how the continuation of the curves of Fig. 13 would appear for values of $2\pi x/\lambda$ greater than 7, since both curves are given with good accuracy in this range by their respective asymptotic representations. Thus Φ will be very closely the same as $\pi \sin(2\pi x/\lambda+\pi/4)$—that is, the exact solution would be one having an almost constant amplitude equal to π—while the amplitude of $\bar{\Phi}$ decreases like $1/(2\pi x/\lambda)^{-1/4}$ (cf. (7.19)). If, therefore, we were to compare the amplitudes at $2\pi x/\lambda=24$, the amplitude of $\bar{\Phi}$ would be about 40 per cent less than that of Φ, while at $2\pi x/\lambda=12$ the error would be about 20 per cent. On the other hand, it should be stated that at $2\pi x/\lambda=7$ the depth of the water for the 6° slope is somewhat less than 1/8 of the wave length. Thus the shallow water theory might be expected to yield fairly accurate results from this point in toward shore, but it could not be expected to do so for $2\pi x/\lambda=12$, much less for $2\pi x/\lambda=24$, on the basis of our discussion above for the case of water of uniform depth.

It is of some interest to compare the phase velocity c furnished by the two theories for $2\pi x/\lambda<7$. This was done by calculating the position of the zeros and the maxima and minima for a series of closely-spaced values of t; the velocities were then obtained by taking difference quotients. The results of such calculations for $2\pi x/\lambda \leqq 7$ are shown in Fig. 14. The asymptotic value for c as furnished by the exact theory is indicated. Up to a distance of about a wave length from shore the two theories yield the same phase velocity, but from this point on the phase velocity predicted by the shallow water theory is too high and becomes more and more inaccurate as the distance from shore (and therefore also the depth of the water) increases. At a distance of about 3 wave lengths from shore the phase velocity as given by the shallow water theory is in error by 10 per cent.

We remark, finally, that the curves of Fig. 14 can not be distinguished (to the scale used there) from the curves

$$c = \sqrt{gh}, \quad \text{and} \quad c = \sqrt{\frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda}} \tag{8.11}$$

in which h is the depth corresponding to any given x and λ is the wave length at ∞. In other words, the phase velocities actually computed by using the two theories for $2\pi x/\lambda \leqq 7$ are practically identical in each of the two theories with those which would be obtained by calculating c for each value of x through use of the corresponding theory applied to water of the correct *uniform* depth.

9. A problem in three-dimensional wave motion. Except for the present section, we consider in this paper only motions which are the same in all planes parallel to an $x-y$-plane, and which therefore can be treated by using functions of a single complex variable. (The exact linear theory is, of course, in question here.) For motions which depend essentially upon three space variables it is not possible to make use of complex functions, but it is possible to extend the basic idea of the method used in the

two-dimensional cases to surface wave problems in three dimensions. In this section we illustrate the method by treating the problem of progressing waves in an infinite ocean bounded on one side by a vertical cliff—in other words, the same problem as that of Sec. 4 except that we no longer require the waves to move with their crests parallel to the shore line.

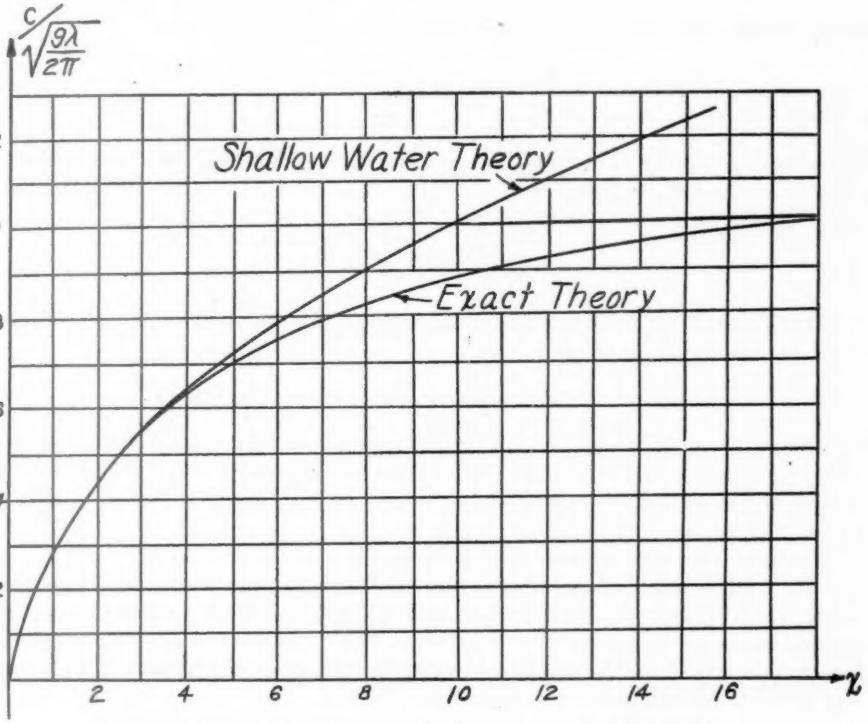

FIG. 14. Phase velocity for progressing waves over a 6° sloping beach

x = distance from shore/$(\lambda/2\pi)$

λ = wavelength at ∞ (exact theory)

c = phase velocity.

We seek solutions $\Phi(x, y, z; t)$ of $\nabla^2_{(x,y,z)}\Phi = 0$ in the region $x \geqq 0$, $y \leqq 0$, $-\infty < z < \infty$ with the y-axis taken normal to the undisturbed free surface of the water and the z-axis[26] taken along the "shore," i.e. at the water line on the vertical cliff $x=0$. Progressing waves moving toward shore are to be found such that the wave crests (or other curves of constant phase) at large distances from shore tend to a straight line which makes an arbitrary angle with the shore line. For this purpose we seek solutions of the form

$$\Phi(x, y, z; t) = e^{i(t+kz+\beta_1)}\varphi(x, y) \tag{9.1}$$

[26] It has already been pointed out that functions of a complex variable are not used in this section, so that the reintroduction of the letter z to represent a space coordinate should cause no confusion with the use of z as a complex variable in earlier sections.

that is, solutions in which periodic factors in both z and t are split off. Of course, the function $\varphi(x, y)$ is not, as in our previous cases, a potential function; instead, it satisfies the differential equation

$$\nabla^2_{(x,y)}\varphi - k^2\varphi = 0, \tag{9.2}$$

as one readily sees. The free surface condition is taken in the form

$$\frac{\partial\varphi}{\partial y} - \varphi = 0 \quad \text{for} \quad y = 0. \tag{9.3}$$

which implies that the dimensionless space and time variables of Sec. 3 (including now z as well as x and y) are assumed at the outset. The condition at the cliff, is, of course,

$$\frac{\partial\varphi}{\partial x} = 0 \quad \text{for} \quad x = 0. \tag{9.4}$$

At the origin $x=0$, $y=0$ (i.e. at the shore line on the cliff) we require, as in former cases, that φ should be of the form

$$\varphi = \bar{\varphi} \log r + \bar{\bar{\varphi}}, \qquad r \ll 1, \tag{9.5}$$

for sufficiently small values of $r = (x^2+y^2)^{1/2}$, with $\bar{\varphi}$ and $\bar{\bar{\varphi}}$ certain bounded functions with bounded first and second derivatives in a neighborhood of the origin. The functions $\bar{\varphi}$ and $\bar{\bar{\varphi}}$ should be considered at present as certain given functions; later on, they will be chosen specifically.

For large values of r we wish to have $\Phi(x, y, z, t)$ behave like $e^{y}e^{i(t+kz+\alpha x+\beta)}$ with $k^2+\alpha^2=1$ but k and α otherwise arbitrary constants, so that progressing waves tending to an arbitrary plane wave at ∞ can be obtained. This requires that $\varphi(x, y)$ should behave at ∞ like $e^{y}e^{i(\alpha x+\beta_2)}$ because of (9.1). However, it is no more necessary here than it was in our former cases to require that φ should behave in this specific way at ∞; it suffices in fact to require that

$$|\varphi| + |\varphi_x| + |\varphi_{xy}| < M \quad \text{for} \quad r > R_0, \tag{9.6}$$

i.e. that φ and the two derivatives of φ occurring in (9.6) should be uniformly bounded at ∞. As we shall see, this requirement leads to solutions of the desired type.

We proceed to solve the boundary value problem formulated in equations (9.2) to (9.6). The procedure we follow is analogous to that used in the former cases in every respect. To begin with, we observe that

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} - 1\right)\varphi = 0 \quad \text{for both } x = 0 \quad \text{and} \quad y = 0, \tag{9.7}$$

because of the special form of the linear operator on the left hand side together with the fact that (9.3) and (9.4) are to be satisfied. A function $\psi(x, y)$ is introduced by the relation

$$\psi = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} - 1\right)\varphi. \tag{9.8}$$

The essential point of our method is that the function ψ is determined uniquely within an arbitrary factor if our function φ, having the properties postulated, exists. Furthermore, ψ can then be given explicitly without difficulty. The properties of ψ are as follows.

1) ψ satisfies the same differential equation as φ, i.e. equation (9.2), as one sees from the definition (9.8) of ψ.
2) ψ is regular in the quadrant $x>0$, $y<0$ and vanishes, in view of (9.7), on $x=0$, $y<0$ and $y=0$, $x>0$. Hence ψ can be continued over the boundaries by the reflection to yield a continuous and single valued function having continuous second derivatives ψ_{xx} and ψ_{yy} (as one can readily see since $\nabla^2\psi-k^2\psi=0$, and $\psi=0$ on the boundaries) in the entire x, y-plane with the exception of the origin. (Here we use the fact that our domain is a sector of angle $\pi/2$.)
3) At the origin, ψ has a possible singularity which is of the form $\bar{\psi}(x, y)/r^2$, with $\bar{\psi}$ regular, as one can see from (9.5) and (9.8). This statement clearly holds for the function ψ when it has been extended by reflection to a full neighborhood of the origin.
4) The condition (9.6) on φ clearly yields for ψ the condition that ψ is uniformly bounded at ∞ after ψ has been extended to the whole plane.

Thus ψ is a solution of $\nabla^2\psi-k^2\psi=0$ in the entire plane which is uniformly bounded at ∞. At the origin $\psi=\bar{\psi}/r^2$ with $\bar{\psi}$ a certain given regular function ($\bar{\psi}=0$ not excluded). In addition, $\psi=0$ on the entire x and y axes. It is not difficult to prove that the solution of this problem is unique.[27]

On the other hand, a solution ψ of the problem for a special function[28] $\bar{\psi}$ characterizing the singularity at the origin is readily given in polar coordinates (r, θ); it is

$$\psi(x, y) = AiH_2^{(1)}(ikr)\sin 2\theta, \qquad r = \sqrt{x^2+y^2}, \qquad 0 \leqq k \leqq 1, \tag{9.9}$$

in which $H_2^{(1)}$ is the Hankel function of order two which tends to zero as $r\to\infty$, and A is any real constant. The function ψ has real values when r is real. (The notation given in Jahnke-Emde, Tables of Functions, is used.) One can readily verify that this function really does satisfy all conditions imposed on ψ. For our purposes it is of advantage to write the solution ψ in the following form:

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[27] One way to do so is the following: The difference, Ψ, of two solutions would have all of the properties of ψ except that it would be regular at the origin. If Ψ_0 is the value of Ψ at any point (x_0, y_0) in the plane, then it is known (see, for example, Courant, Hilbert, Methoden d. Math. Phys., Bd. II, S.261) that the mean value formula

$$\Psi_0\cdot J_0(ikR) = M$$

holds. Here M is the mean value of Ψ over any circle of radius R and center at (x_0, y_0). The function J_0 is the regular Bessel function of order zero. If R is chosen large enough M remains less than a certain constant since Ψ is uniformly bounded at ∞. On the other hand, $J_0(ikR)$ behaves for large R like $e^{kR}(2\pi kR)^{-1/2}$ (see Jahnke-Emde, Tables of Functions, p. 138) and hence as $R\to\infty$, Ψ_0 would tend to zero. But since Ψ_0 is independent of R it follows that Ψ_0 is zero at any arbitrary point (x_0, y_0). Thus $\Psi\equiv 0$, and the uniqueness of the function ψ is proved.

[28] Our uniqueness theorem is less general in the present case than in the earlier cases since we prescribe the singularity at the origin so specifically in the present case.

$$\psi = Ai\frac{\partial^2}{\partial x \partial y} H_0^{(1)}(ikr), \qquad r = \sqrt{x^2 + y^2}, \tag{9.10}$$

in which A is any real constant and $H_0^{(1)}$ is the Hankel function of order zero which is bounded as $r \to \infty$. It is readily verified that this solution differs from that given by (9.9) only by a constant multiplier; one can do so, for example, by using the well-known identities involving the derivatives of Bessel functions of different orders.

Once ψ is determined we may write (9.8) in the form

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} - 1\right)\varphi = Ai\frac{\partial^2}{\partial x \partial y} H_0^{(1)}(ikr), \qquad A \text{ arbitrary.} \tag{9.11}$$

This means that our function φ, if it exists, must satisfy (9.11) as well as (9.2). By integration of (9.11) it turns out that we are able to determine φ explicitly without great difficulty on account of the simple form of the left hand side of (9.11).[29] This we proceed to do.

Integration of both sides of (9.11) with respect to x leads to

$$\left(\frac{\partial}{\partial y} - 1\right)\varphi = Ai\frac{\partial}{\partial y} H_0^{(1)}(ikr) + g(y), \tag{9.12}$$

in which $g(y)$ is an arbitrary function. But $g(y)$ must satisfy (9.2), since all other terms in (9.12) satisfy it. Hence $d^2g/dy^2 - k^2 g = 0$. In addition $g(0) = 0$, since the other terms in (9.12) vanish for $y = 0$ because of (9.3) and the fact that $\partial/\partial y H_0^{(1)} = ik(y/r)dH_0^{(1)}/dr$. Finally, $g(y)$ is bounded as $y \to -\infty$ because of condition (9.6) and the fact that $\partial H_0^{(1)}/\partial y$ tends to zero as $r \to \infty$. The function $g(y)$ is therefore readily seen to be identically zero. By integration of (9.12) we obtain (after setting $g(y) = 0$):

$$\varphi = Aie^y \int_{+\infty}^{y} e^{-t} \frac{\partial}{\partial t}\left[H_0^{(1)}(ik\sqrt{x^2 + t^2})\right]dt + B(x)e^y. \tag{9.13}$$

The function $B(x)$ and the real constant A are arbitrary. The integral converges, since $\partial/\partial t(H_0^{(1)})$ dies out exponentially as $t \to \infty$.

We shall see that two solutions $\varphi_1(x, y)$ and $\varphi_2(x, y)$ satisfying all conditions of our problem can be obtained from (9.13) by taking $A = 0$ in one case and $A \neq 0$ in the other case, and that these solutions will be 90° "out of phase" at ∞. (This is exactly analogous to the behavior of the solutions in our previous cases.)

Consider first the case $A = 0$. The function φ given by (9.13) satisfies (9.2) only if

$$\frac{d^2B(x)}{dx^2} + (1 - k^2)B(x) = 0. \tag{9.14}$$

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[29] It can now be seen how the differential equation corresponding to (9.11) could be obtained for the case of waves coming in toward shore over a uniformly sloping beach at inclination $\pi/2n$ to the horizontal: The left hand side would be a differential operator on ϕ of order $2n$, which could be obtained from (5.9) by going over to real operators. The right hand side would be essentially the Hankel function $H_{2n}^{(1)}(ikr)$ of order $2n$ multiplied by $\sin 2n\theta$. In fact, the entire discussion of sec. 5) for the angles $\pi/2n$ could be generalized to yield three-dimensional progressing wave solutions by proceeding in the manner of the present section.

It is important to recall that $k^2<1$. The boundary condition $\varphi_x=0$ for $x=0$ requires that $B_x(0)=0$. The condition $\varphi_y-\varphi=0$ for $y=0$ is automatically satisfied because of (9.12) and $g(y)\equiv 0$. Hence $B(x)=A_1\cos\sqrt{1-k^2}x$, with A_1 arbitrary, and the solution $\varphi_1(x, y)$ is

$$\varphi_1(x, y) = A_1e^y \cos \sqrt{1 - k^2}\, x. \tag{9.15}$$

This leads to solutions Φ_1 in the form of standing waves,[30] as follows:

$$\Phi_1(x, y, z, t) = A_1e^{it}e^y \cos \sqrt{1 - k^2}\, x.\begin{Bmatrix}\cos kz\\ \sin kz\end{Bmatrix} \tag{9.15$'$}$$

for $k^2<1$. If $k=1$, the solution Φ_1 given by (9.15′) is valid, although it could not be obtained by our process since $B(x)$ would be identically zero then.

As we have already stated, we obtain solutions $\varphi_2(x, y)$ from (9.13) for $A\neq 0$ which behave for large x like $\sin\sqrt{1-k^2}x$ rather than like $\cos\sqrt{1-k^2}x$, and with these two types of solutions progressing waves approaching an arbitrary plane wave at ∞ can be constructed by superposition.

We begin by showing that (9.2) is satisfied for all $x>0$, $y<0$ by φ as given in (9.13) with $A\neq 0$, provided only that $B(x)$ satisfies (9.14). Since $x>0$, it is permissible to differentiate under the integral sign in (9.13), even though t takes on the value zero (since the upper limit y is negative). By differentiating we obtain

$$\nabla^2\varphi - k^2\varphi = Ai\left\{e^y\int_\infty^y e^{-t}\frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial x^2} + (1 - k^2)\right]H_0^{(1)}dt \right.$$
$$\left. + \frac{\partial H_0^{(1)}}{\partial y} + \frac{\partial^2 H_0^{(1)}}{\partial y^2}\right\} + B''(x) + (1 - k^2)B. \tag{9.16}$$

Since $H_0^{(1)}$ is a solution of (9.2) the operator $(\partial^2/\partial x^2-k^2)$ occurring under the integral sign can be replaced by $-\partial^2/\partial y^2$ and hence the integral can be written in the form

$$\int_\infty^y e^{-t}\left[-\frac{\partial^3}{\partial t^3} + \frac{\partial}{\partial t}\right]H_0^{(1)}(ikr)dt.$$

We introduce the following notation

$$I_m(x, y) = e^y\int_\infty^y e^{-t}\frac{\partial^m}{\partial t^m}H_0^{(1)}(ikr)dt,$$

and obtain through two integrations by parts the relation

$$I_m(x, y) = \left[\frac{\partial^{m-1}}{\partial y^{m-1}} + \frac{\partial^{m-2}}{\partial y^{m-2}}\right]\cdot H_0^{(1)} + e^y\int_{+\infty}^y e^{-t}\frac{\partial^{m-2}H_0^{(1)}}{\partial t^{m-2}}dt,$$

in which we have made use of the fact that the boundary terms are zero at the lower limit $+\infty$, since all derivatives of $H_0^{(1)}(ikr)$ tend to zero as $r\to+\infty$. The integral of interest to us is given obviously by I_1-I_3 and this in turn is given by

[30] The standing wave solutions of this type (but not of the type with a singularity) for beaches sloping at angles $\pi/2n$ were obtained by Hanson [3] by a quite different method.

$$- I_3 + I_1 = - \frac{\partial^2 H_0^{(1)}}{\partial y^2} - \frac{\partial H_0^{(1)}}{\partial y} - e^y \int_\infty^y e^{-t} \frac{\partial H_0^{(1)}}{\partial t} dt$$

$$+ e^y \int_\infty^y e^{-t} \frac{\partial H_0^{(1)}}{\partial t} dt = - \frac{\partial^2 H_0^{(1)}}{\partial y^2} - \frac{\partial H_0^{(1)}}{\partial y}$$

by use of the above relations for I_m. Hence the quantity in brackets in (9.16) is identically zero—in other words the term containing the integral on the right hand side of (9.13) is a solution of (9.2). Hence φ is a solution of (9.2) in the case $A \neq 0$ if $B(x)$ satisfies (9.14). Since (9.12) holds and $g(y) \equiv 0$ it follows that the free surface condition (9.3) is satisfied by φ in view of the fact that $\partial H_0^{(1)}(ikr)/\partial y = 0$ for $y = 0$.

We have still to show that a solution $B(x)$ of (9.14) can be chosen so that $\varphi_x = 0$ for $x = 0$, and that φ has the desired behavior for large values of r. Actually, these two things go hand in hand, just as in former cases. An integration by parts in (9.13) yields the following for φ:

$$\varphi = Aie^y \int_\infty^y e^{-t} H_0^{(1)}(ik\sqrt{x^2 + t^2})dt + AiH_0^{(1)}(ik\sqrt{x^2 + y^2}) + B(x)e^y, \qquad (9.17)$$

provided that $x > 0$. It should be recalled that the upper limit y of the integral is negative; thus the integrand has a singularity for $x = 0$ since $t = 0$ is included in the interval of integration and $H_0^{(1)}(ikr)$ is singular for $r = 0$. We shall show that $\lim_{x \to 1} \partial\psi/\partial x = 0$ provided that $B_x(0) = -2A \neq 0$. We have, for $x > 0$ and $y < 0$:

$$\frac{\partial \varphi}{\partial x} = Aie^y \int_\infty^y e^{-t} \frac{\partial}{\partial x} [H_0^{(1)}(ik\sqrt{x^2 + t^2})]dt + Ai \frac{\partial}{\partial x} [H_0^{(1)}(ik\sqrt{x^2 + y^2})] + B_x(x)e^y.$$

The second term on the right hand side is readily seen to approach zero as $x \to 0$ since this term can be written as the product of x and a factor which is bounded for $y < 0$. For the same reason it is clear that the only contribution furnished by the integral in the limit as $x \to 0$ arises from a neighborhood of $t = 0$ since the factor x may be taken outside of the integral sign. We therefore consider the limit

$$\lim_{x \to 0} \int_\epsilon^{-\epsilon} e^{-t} \frac{\partial}{\partial x} [iH_0^{(1)}(ik\sqrt{x^2 + t^2})dt, \qquad \epsilon > 0.$$

The function $iH_0^{(1)}(ikr)$ has the following development valid near $r = 0$:

$$iH_0^{(1)}(ikr) = - \frac{2}{\pi} [J_0(ikr) \log r + p(r)]$$

in which $p(r)$ represents a convergent power series containing only even powers of r (including a zero power), and J_0 is the regular Bessel function with the following development

$$J_0(ikr) = 1 + \frac{(kr)^2}{2^2} + \cdots .$$

It follows that

$$\frac{\partial}{\partial x}\left[iH_0^{(1)}(ikr)\right] = -\frac{2}{\pi}\left[\frac{x}{r^2}J_0(ikr) + J_0'(ikr)\frac{x}{r}\log r + xg(r)\right]$$

$$= -\frac{2}{\pi}\left[\frac{x}{r^2}J_0(ikr) + \tfrac{1}{2}kx\log r + xg(r)\right]$$

in which $g(r)=(1/r)dp/dr$ is bounded as $x\to 0$ since $y<0$. The contribution of our integral in the limit is therefore easily seen to be given by

$$\lim_{x\to 0} -\frac{2}{\pi}\int_{\epsilon}^{-\epsilon} e^{-t}\cdot\frac{x}{x^2+t^2}dt = \lim_{x\to 0} -\frac{2}{\pi}\int_{\epsilon}^{-\epsilon}\frac{x}{x^2+t^2}dt.$$

By introducing $u=t/x$ as new integration variable and passing to the limit we may write

$$\lim_{x\to 0} -\frac{2}{\pi}\int_{\epsilon}^{-\epsilon}\frac{x}{x^2+t^2}dt = -\frac{2}{\pi}\int_{\infty}^{-\infty}\frac{du}{1+u^2} = 2.$$

It therefore follows that $\lim_{x\to 0}\partial\varphi/\partial x=0$ provided that

$$B_x(0) = -2A. \tag{9.18}$$

The function $B(x)$ which satisfies this condition and the differential equation (9.14) is

$$B(x) = -\frac{2A}{\sqrt{1-k^2}}\sin\sqrt{1-k^2}\,x. \tag{9.19}$$

Since $H_0^{(1)}(ikr)$ dies out exponentially as $r\to\infty$ it follows that the solution φ given by (9.17) with $B(x)$ defined by (9.19) behaves at ∞ like $e^y\sin\left[(1-k^2)^{1/2}x\right]$.

A solution φ_2 of our problem which is out of phase with φ_1 (cf. (9.15)) is therefore given by

$$\varphi_2(x, y) = A_2\Big[ie^y\int_{\infty}^{y}e^{-t}H_0^{(1)}(ik\sqrt{x^2+t^2})dt$$

$$+ iH_0^{(1)}(ik\sqrt{x^2+y^2}) - \frac{2e^y}{\sqrt{1-k^2}}\sin\sqrt{1-k^2}\,x\Big], \tag{9.20}$$

with A_2 an arbitrary real constant. A standing wave solution Φ_2 is then given by

$$\Phi_2 = A_2e^{it}\varphi_2(x, y)\cdot\begin{Bmatrix}\cos kz\\ \sin kz\end{Bmatrix}. \tag{9.20'}$$

By taking appropriate values of k progressing waves tending at ∞ to any arbitrary plane wave solution for water of infinite depth can be obtained by forming proper linear combinations of solutions of the type (9.15′) and (9.20′). For a progressing wave traveling toward shore, for example, we might write

$$\begin{aligned}\Phi(x, y, z; t) = {}& A\left[\varphi_1(x, y)\cos kz + \frac{\sqrt{1-k^2}}{2}\varphi_2(x, y)\sin kz\right]\cos t\\ & - A\left[\varphi_1(x, y)\sin kz - \frac{\sqrt{1-k^2}}{2}\varphi_2(x, y)\cos kz\right]\sin t\end{aligned} \tag{9.21}$$

in which A_1 and A_2 in (9.15) and (9.20) are both taken equal to A. The solution (9.21) behaves at ∞ like $Ae^y \cos(\sqrt{1-k^2}x+kz+t)$ as one can readily verify by making use of the asymptotic behavior of $\varphi_1(x, y)$ and $\varphi_2(x, y)$.[31]

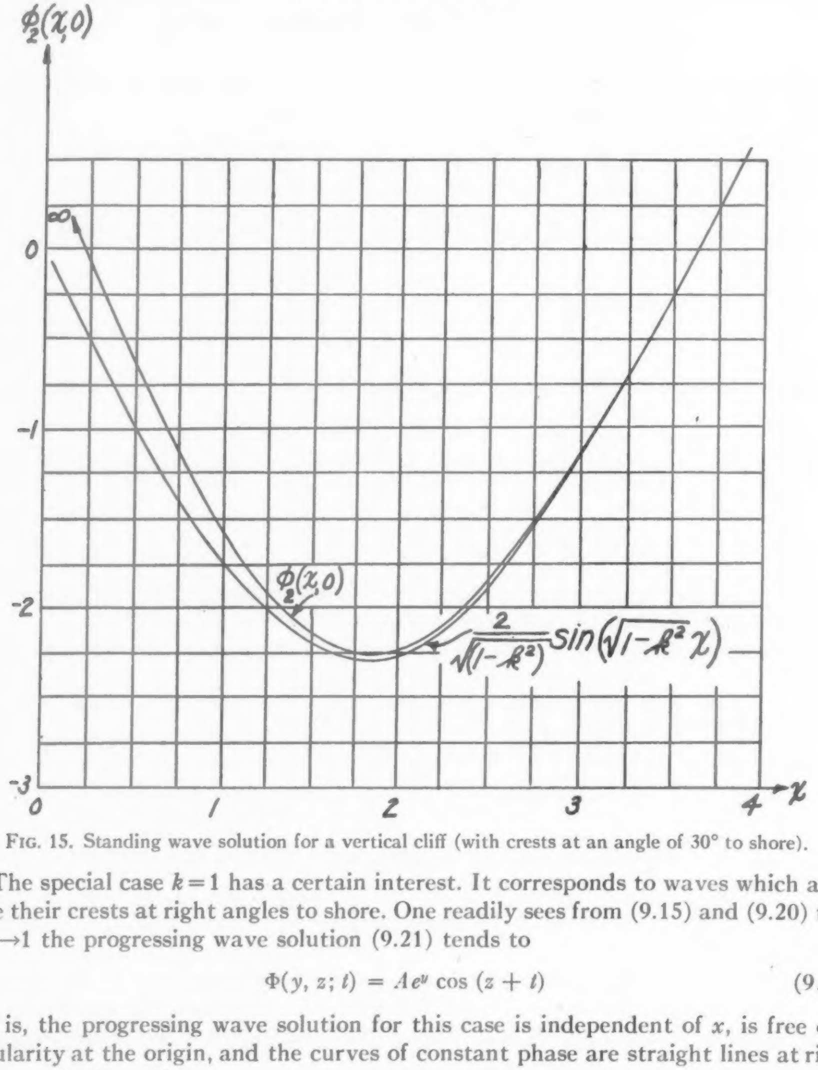

FIG. 15. Standing wave solution for a vertical cliff (with crests at an angle of 30° to shore).

The special case $k=1$ has a certain interest. It corresponds to waves which at ∞ have their crests at right angles to shore. One readily sees from (9.15) and (9.20) that as $k \to 1$ the progressing wave solution (9.21) tends to

$$\Phi(y, z; t) = Ae^y \cos(z + t) \tag{9.22}$$

that is, the progressing wave solution for this case is independent of x, is free of a singularity at the origin, and the curves of constant phase are straight lines at right angles to the shore line.

The progressing wave solution (9.21) has been discussed numerically for $k=1/2$, i.e. for the case in which the wave crests tend at ∞ to a straight line inclined at 30° to the shore line. The function $\varphi_2(x, 0)$ is plotted in Fig. 15. With the aid of these val-

[31] We remark once more that the original space and time variables can be reintroduced simply by replacing x, y, z by mx, my, mz and t by σt (cf. Sec. 3).

ues the contours for $\Phi(x, 0, z; 0)$, i.e. for the surface of the water at time $t=0$, were calculated and are given in Fig. 16. The water surface is shown between a pair of successive "nodes" of Φ, that is, curves for which $\Phi=0$. These curves go into the z-axis (the shore line) under zero angle, as do all other contour lines. This is seen at once from their equation (cf. (9.21) with $t=0$).

$$\varphi_1(x, 0) \cos kz + \frac{\sqrt{1-k^2}}{2} \varphi_2(x, 0) \sin kz = \eta = \text{const.} \tag{9.23}$$

Since $\varphi_2 \to \infty$ as $x \to 0$ while φ_1 remains bounded, it is clear that $\sin kz$ must approach

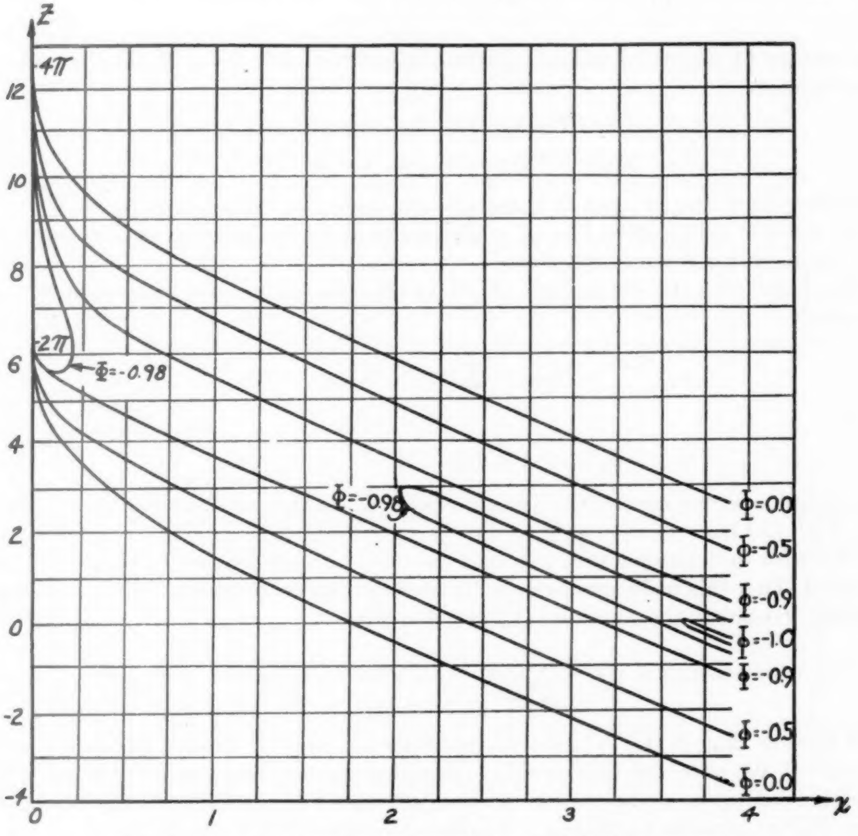

FIG. 16. Level lines for a wave approaching a vertical cliff at an angle.

zero as $x \to 0$ on any such curve. That the contours are all tangent to the z-axis at the points $z=2\pi n$, n an integer, is also readily seen. It is interesting to observe that the height of the wave crest is lower at some points near to the cliff than it is at ∞ (where the value is minus one), so that a saddle point occurs. We observe also that a contour for $\eta=-1$ occurs at the right hand edge of Fig. 16. It is believed that the numerical calculations are sufficiently accurate to guarantee the existence of such a contour for

$\eta=-1$; if so, then this contour is likely to be a closed curve, since $\eta \to 1$ at ∞. It may be that the wave crest is a ridge with a number of saddle points.[32]

Appendix I. Solution of the complex differential equation in the general case

In this appendix we give some of the details of the methods used to obtain explicit solutions of the differential equation (5.11) for the analytic function $f(z)$ which satisfies the boundary conditions (5.1) and (5.2). The differential equation (5.11) is

$$\prod_{k=1}^{n} (e^{-i(k/n+1/2)\pi}D - 1)\cdot f = A \frac{i}{z^n}, \qquad A \text{ real.} \tag{I.1}$$

The symbol $\prod$ means, as usual, a continued product, and $D=d/dz$. The boundary conditions are

$$\Re e(iD - 1)f = 0 \quad \text{for} \quad z \text{ real, and} \tag{I.2}$$

$$\Re e(ie^{-i\pi/2n}D)f = 0 \quad \text{on} \quad z = re^{-i\pi/2n}, \tag{I.3}$$

in conformity with (5.1) and (5.2) for a beach sloping at the angle $\pi/2n$.

In order to obtain all phases at ∞ it is sufficient to obtain solutions $f_1(z)$ and $f_2(z)$ for $A=0$ and for $A\neq 0$, respectively.

We begin with the simpler case $A=0$. In this case the general solution of (I.1) is obviously

$$f_1(z) = \sum_{k=1}^{n} c_k e^{z\beta_k}, \quad \text{with} \tag{I.4}$$

with

$$\beta_k = e^{i\pi(k/n+1/2)}, \qquad k = 1, 2, \cdots, n. \tag{I.5}$$

The c_k are arbitrary constants. (The quantities β_k are the same as those used in Secton, 5, cf. (5.13).)

We wish to determine the c_k so that the boundary conditions (I.2) and (I.3) are satisfied. That this really can be done for all points on these lines is far from obvious a priori. To satisfy the condition (I.2) we first write

$$H(z) = (iD - 1)f_1 = \sum_{k=1}^{n} (i\beta_k - 1)c_k e^{z\beta_k} = -\sum_{k=1}^{n} (e^{i\pi k/n} + 1)c_k e^{z\beta_k},$$

and observe that in this relation the coefficient c_n may be chosen arbitrarily since $e^{i\pi}+1=0$. We note also that $\beta_k=\bar{\beta}_{n-k}$, in which the bar over a number denotes the complex conjugate of the number so that $e^{z\beta_k}=\overline{e^{z\beta_{n-k}}}$ if z is real. In order to ensure that $\Re e H(z)$ vanishes for *all real* z, it is therefore necessary to choose the constants c_k in such a way that

$$H(\bar{z}) = -\overline{H(z)}, \quad \text{for } z \text{ real.}$$

[32] It should be pointed out that we are no more able to decide in the present case than we were in the two-dimensional cases whether the waves are reflected back to infinity from the shore, and if so to what extent. Our numerical solution was obtained on the assumption that no reflection takes place, which is probably not well justified for the case of a vertical cliff, but would be for a beach of small slope.

To satisfy the above condition, we equate the pair of terms

$$(e^{i\pi k/n}+1)c_k e^{z\beta_k}; \quad -\overline{[e^{i\pi(n-k)/n}+1]c_{n-k}e^{z\beta_{n-k}}}, \qquad (k=1,2,\cdots,n-1)$$

Since $e^{z\beta_k}$ and $\overline{e^{z\beta_{n-k}}}$ are conjugate for z real, we need only require

$$(e^{i\pi k/n}+1)c_k = -\overline{[e^{i\pi(n-k)/n}+1]}\bar{c}_{n-k}, \quad \text{or} \quad c_k = \frac{[e^{i\pi k/n}-1]}{[e^{i\pi k/n}+1]}\bar{c}_{n-k}.$$

Therefore, if the c_k are chosen so that

$$c_k = i\tan(\pi k/2n)\bar{c}_{n-k}, \qquad (k=1,2,\cdots,n-1) \tag{I.6}$$

the boundary condition (I.2) will be satisfied.

The condition at the bottom requires a similar analysis of the expression

$$K(z) = e^{i\pi(n-1)/2n}Df_1 = \sum_{k=1}^{n} e^{i\pi(n-1)/2n}\beta_k c_k e^{z\beta_k} \quad \text{for} \quad z = re^{-i\pi/2n}$$

or, upon insertion of the special values of z:

$$K(re^{-i\pi/2n}) = -\sum_{k=1}^{n} e^{i\pi(2k-1)/2n}c_k e^{re^{i\pi(2k-1+n)/2n}} = L(r).$$

The real part of $L(r)$ should vanish for all $r>0$. We observe that $e^{re^{i\pi(2k-1+n)/2n}}$ is conjugate to $e^{re^{i\pi[2(n+1-k)-1+n]/2n}}$. Hence our requirement that $L(r)$ be real leads to

$$e^{i\pi(2k-1)/2n}c_k = -\overline{e^{i\pi(2(n+1-k)-1+n)/2n}c_{n+1-k}}$$

or, as one readily sees

$$c_k = \bar{c}_{n+1-k}, \quad \text{for} \quad k=1,2,\cdots,n. \tag{I.7}$$

Thus, in order to satisfy the boundary conditions at the free surface as well as at the bottom we must impose on the c_k the conditions

$$\begin{aligned} c_k &= i\bar{c}_{n-k}\tan(\pi k/2n), & k&=1,2,\cdots,n-1 \\ c_k &= \bar{c}_{n-k+1}, & k&=1,2,\cdots,n. \end{aligned} \tag{I.8}$$

We proceed to show that these relations can be satisfied by a set of values c_k which are uniquely determined within a real multiplying factor.

From (I.8) we easily obtain the following recurrence relation by taking conjugates and eliminating $\bar{c}_k$

$$c_{n-k} = ic_{n-k+1}\cot\frac{\pi k}{2n}, \qquad k=1,2,\cdots,n-1,$$

which may also be expressed in the form

$$c_{n-k} = (i)^k c_n \cot\frac{\pi}{2n}\cot\frac{2\pi}{2n}\cdots\cot\frac{k\pi}{2n}, \quad \text{for} \quad k=1,2,\cdots,n-1. \tag{I.9}$$

For $k=n$ we have the additional relation

$$c_1 = \bar{c}_n. \tag{I.10}$$

If we set $k = n-1$ in (I.9) the cotangents cancel each other and the relation $c_1 = (i)^{n-1}c_n$ results; this relation is easily seen to be compatible with (I.10) only if c_n is given as follows:

$$c_n = re^{-i\pi(n-1)/4} \tag{I.11}$$

in which r is any real number. In order to fix the c_k in such a way as to satisfy the boundary conditions it is therefore only necessary to choose c_n in accordance with (I.11) and calculate the remaining quantities by using (I.9). The following somewhat more convenient form might also be used to calculate the c_k:

$$\begin{aligned} c_k &= re^{i\pi[(n+1)/4-k/2]} \cot\frac{\pi}{2n} \cdots \cot\frac{\pi(k-1)}{2n}, \qquad k = 2, 3, \cdots, n, \\ c_1 &= \bar{c}_n \end{aligned} \tag{I.12}$$

the first relation resulting through combining the first of (I.8) with (I.9) and noting afterwards that the relation holds for $k = n$ since the product of the cotangents has the value one in this case and c_k for $k = n$ thus has the value given by (I.11).

We turn next to the case $A \neq 0$ and, in fact, set $A = 1$ for the purpose of the present investigation. It is clear that the results for any other value of A are obtained simply by multiplication by A. A solution $f_2(z)$ of the non-homogeneous equations can be obtained without difficulty, though the calculations are somewhat laborious. Instead of proceeding constructively we prefer to give the solution and then verify that it satisfies all conditions, particularly the boundary conditions. The solution $f_2(z)$ is*

$$f_2(z) = \sum_{k=1}^{n} a_k \left\{ e^{z\beta_k} \int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\, dt - \pi i e^{z\beta_k} \right\}. \tag{I.13}$$

In this solution the β_k are given by (I.5) and the a_k turn out to be multiples of the c_k:

$$a_k = c_k/(n-1)!\sqrt{n}. \tag{I.14}$$

These values of the a_k are of course required in order that $f_2(z)$ should satisfy the inhomogeneous equation. The path of integration for the complex integrals in (I.13) has already been given by Fig. 2 of Sec. 4. The path of integration comes from $+\infty$ along the real axis, then goes along a circular arc with center at the origin (leaving the origin to the left), and finally along a ray from the origin to the point $z\beta_k$. Since z lies in the sector for which $-\pi/2n \leq \arg z \leq 0$ and the β_k are given by (I.5) it follows that $z\beta_k$ always lies in the left half-plane. We consider first the condition (I.2). We write

$$(iD - 1)f_2 = \sum_{k=1}^{n} (i\beta_k - 1)a_k\{A_k\} + \frac{i}{z}\sum_{k=1}^{n} a_k = M(z) + \frac{i}{z}\sum_{k=1}^{n} a_k$$

in which $\{A_k\}$ has an obvious significance. Since $\sum_{k=1}^{n} a_k$ is real, (as one sees from (I.7) and (I.14)) the last term is pure imaginary when z is real. We must verify that the a_k are so chosen that the real part of the remaining terms, $M(z)$, vanishes for z real.

* Compare with (5.15) of Sec. 5.

As with the similar problem with the c_k above, we must verify that $M(\bar{z}) = -\overline{M(z)}$ for z real. In this case it is the terms

$$(i\beta_k - 1)a_k\{A_k\} \quad \text{and} \quad (i\beta_{n-k} - 1)a_{n-k}\{A_{n-k}\}$$

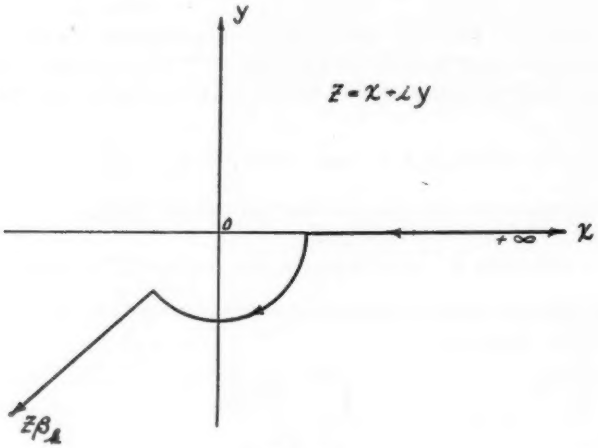

Fig. 17

for which the real parts cancel on addition. By evaluating the residue of $\oint e^{-t}dt/t$ at the origin we find readily that

$$a_k\{A_k\} = a_k\left\{e^{z\beta_k}\,{}_C\!\int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\,dt - \pi i e^{z\beta_k}\right\} = a_k\left\{e^{z\beta_k}\,{}_{\overline{C}}\!\int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\,dt + \pi i e^{z\beta_k}\right\}$$

in which the integral on the right hand side of this equation is taken along the path $\overline{C}$ shown in Fig. 17, while the integral on the left is taken over the path C shown in Fig. 2. The origin is kept to the right on $\overline{C}$ instead of to the left. We observe that the exponential terms $e^{z\beta_k}$ and $e^{z\beta_{n-k}}$ are complex conjugates for z real, so that $-ie^{-z\beta_k}$ and $+ie^{z\beta_{n-k}}$ are complex conjugates, since the same is true for β_k and β_{n-k}. In addition, the integrals in $\{A_k\}$ and $\{A_{n-k}\}$ are complex conjugates, as one sees from the above relation: if we replace k by $n-k$ in the integral along $\overline{C}$ it is clear that it becomes the conjugate of the integral along the path conjugate to $\overline{C}$, since the integrands are the same but the paths are complex conjugates of each other. Consequently we have only to verify that

$$(i\beta_k - 1)a_k = \frac{1}{-(i\beta_{n-k} - 1)a_{n-k}}\,.$$

But this is the same as the corresponding relation for the c_k given above, and hence is satisfied by the a_k since the a_k are the same as the c_k except for a real multiplying factor.

To check the condition (I.3) at the bottom we consider the expression

$$e^{i\pi(n-1)/2n}Df_2(z) = \sum_{k=1}^{n} e^{i\pi/2}e^{-i\pi/2n}\beta_k a_k\{A_k\} + z^{-1}e^{i\pi/2}e^{-i\pi/2n}\sum_1^n a_k$$

$$= N(z) + z^{-1}e^{i\pi/2}e^{-i\pi/2n}\sum_1^n a_k$$

evaluated for $z = re^{i\pi/2n}$. The last term is pure imaginary for $z = re^{-i\pi/2n}$ since $\sum_1^n a_k$ is real. It can be shown that $\Re e N(z) = 0$ for $z = re^{-i\pi/2n}$ by proceeding in the same way as above, except that the terms should be paired in a slightly different manner. In fact, the terms

$$e^{i\pi/2}e^{-i\pi/2n}\beta_k a_k\{A_k\} \quad \text{and} \quad e^{i\pi/2}e^{-i\pi/2n}\beta_{n+1-k}a_{n+1-k}\{A_{n+1-k}\}$$

are negative conjugates in this case, as one can readily verify.

APPENDIX II: ASYMPTOTIC BEHAVIOR OF $\int_\infty^{z\beta_k}(e^{-t}/t)dt$

In this appendix we prove a number of assertions made in Sec. 5 with regard to the behavior of the integrals

$$\int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\,dt, \tag{II.1}$$

$$\beta_k = e^{i\pi(k/n+1/2)}, \qquad k = 1, 2, \cdots, n \tag{II.2}$$

when $z \to \infty$ in the sector S defined by

$$S\colon\ 0 \geqq \arg z \geqq -\frac{\pi}{2n}\cdot$$

The path of integration is given by Fig. 2 of Sec. 4.

From (II.2) and the definitions of the sector S it follows that

$$\pi/2 \leqq \arg z\beta_k \leqq 3\pi/2. \tag{II.3}$$

It might also be noted that $\arg z\beta_k = 3\pi/2$ only for $k = n$, i.e. for $\beta_k = -i$, and z real. We shall show that the integral (II.1) behaves for z in S and $|z|$ large as follows:

$$\int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\,dt \sim \begin{cases} -\dfrac{e^{-z\beta_k}}{\beta_k}\left(\dfrac{1}{z} + \cdots\right), & \dfrac{\pi}{2} < \arg z\beta_k \leqq \pi, \\ 2\pi i - \dfrac{e^{-z\beta_k}}{\beta_k}\left(\dfrac{1}{z} + \cdots\right), & \pi < \arg z\beta_k \leqq \dfrac{3\pi}{2}, \end{cases} \tag{II.4}$$

in which the dots refer to terms of higher order in $1/z$.

We consider first the case in which $\pi/2 \leqq \arg z\beta_k \leqq \pi$ and begin by integrating twice by parts to obtain

$$\int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\,dt = -\frac{e^{-z\beta_k}}{z\beta_k} + \frac{e^{-z\beta_k}}{(z\beta_k)^2} + 2\int_{\infty}^{z\beta_k} \frac{e^{-t}}{t^3}\,dt. \tag{II.5}$$

We shall show that

$$\left| \int_{\infty}^{z\beta_k} \frac{e^{-t}}{t^3} dt \right| \leq c \left| \frac{e^{-z\beta_k}}{z^2} \right|$$

for z in the sector S, with c a positive number independent of z. Clearly, this would suffice to show that our integral behaves at ∞ like $-e^{-z\beta_k}/z\beta_k$.

We consider the integral $\oint(^{-t}/t^3)dt$ along the closed path indicated in Fig. 17. Since the integral over the closed path vanishes, it is clear that the behavior of our integral as $z\to\infty$ can be reduced to the investigation of the behavior of $\int(e^{-t}/t^3)dt$ over the circular arc PQR as $r\to\infty$. The point P corresponds to $z\beta_k$ of course. Upon setting $t=r(\cos\theta+i\sin\theta)$ the integral becomes

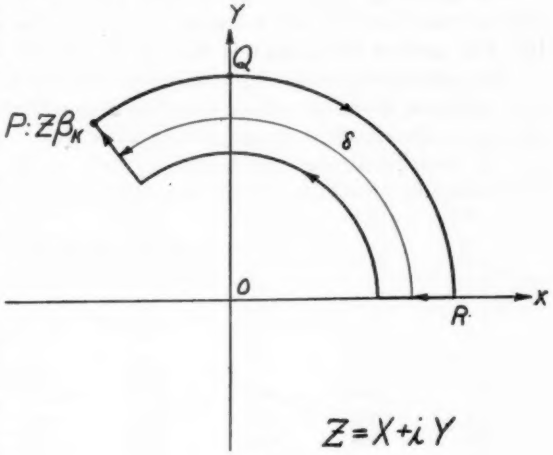

FIG. 18

$$I = i\int_{\delta}^{0} e^{-r\cos\theta}e^{-ri\sin\theta}\cdot\frac{d\theta}{r^2e^{2i\theta}}\cdot \tag{II.6}$$

For $|I|$ we have, obviously

$$|I| \leq 2\left|\int_0^{\delta}\frac{1}{r^2}\cdot e^{-r\cos\theta}d\theta\right|,$$

and we may write

$$|I| \leq 2\frac{e^{-r\cos\delta}}{r^2}\int_0^{\delta}d\theta = 2\delta\frac{e^{-r\cos\delta}}{r^2}, \quad \text{with} \quad \delta = \arg z\beta_k,$$

since $e^{-r\cos\theta}\leq e^{-r\cos\delta}$ for $0\leq\theta\leq\delta\leq\pi$. We observe that

$$|e^{-z\beta_k}/(z\beta_k)^2| = r^{-2}e^{-r}\cos\delta, \qquad r = |z\beta_k|,$$

and this establishes (II.4) for $\pi/2\leq\arg z\beta_k\leq\pi$.

To establish (II.4) for $\arg z\beta_k>\pi$, we write

$$\int_{C\ +\infty}^{z\beta_k}\frac{e^{-t}}{t}dt = 2\pi i + \int_{\bar{C}\ +\infty}^{z\beta_k}\frac{e^{-t}}{t}dt,$$

by evaluating the residue at the origin. The integral over $\bar{C}$ (see Fig. 17) can be treated in the same fashion as above. The only difference is that $\int(e^{-t}/t^3)dt$ is taken over a circular arc such that $\delta\leq\theta\leq 0$ with $-\pi<\delta\leq-\pi/2$. The integral I' for this case corresponding to I in (II.6) is the same as that for I except that δ is replaced by $-\delta$. The inequalities for $|I'|$ are thus exactly the same as for $|I|$ since $\cos\theta$ is an even function and δ lies in the range $\pi<\delta\leq\pi/2$. Thus (II.4) is established in general.

APPENDIX III: TABLE OF $E(z)=e^z\int_z^\infty (e^{-u}/u)du$ for $z=re^{i\beta_k}$; $\beta_k=e^{i\pi(k/15+1/2)}$; $k=1,\cdots,7$

The following table of the function $E(z)=e^z\int e^{-u}du/u$ is an extension of the short table of this function which has been prepared by the Mathematical Tables Project [8]. The path of integration is that of Fig. 2, but taken in the opposite direction.

The calculations were begun with fourteen six-decimal place values of the function $E(z)$ obtained from the above mentioned source, M.T.P. [8]. [We then computed* the rest of the numbers by means of power series expansions along each of the seven rays in the second quadrant defined by $re^{i\beta_k}$; $k=1,\cdots,7$; $1\leqq r\leqq 7$.] The values are believed to be correct to within one unit in the last figure.

Real part of $E(z)$

r \ k	1	2	3	4	5	6	7
1.00	+.2650	+.1731	+.0657	−.0597	−.2063	−.3784	−.5817
.25	.1901	.1013	−.0023	−.1232	−.2644	−.4304	−.6271
.50	.1398	.0551	−.0434	−.1580	−.2912	−.4473	−.6315
.75	.1046	.0242	−.0688	−.1763	−.3006	−.4449	−.6139
2.00	+.0790	+.0031	−.0844	−.1849	−.2999	−.4322	−.5849
.25	.0601	−.0117	−.0939	−.1875	−.2936	−.4140	−.5507
.50	.0457	−.0222	−.0993	−.1864	−.2841	−.3933	−.5149
.75	.0345	−.0296	−.1020	−.1831	−.2730	−.3719	−.4795
3.00	+.0258	−.0348	−.1029	−.1784	−.2612	−.3507	−.4457
.25	.0190	−.0385	−.1026	−.1731	−.2494	−.3304	−.4141
.50	.0135	−.0411	−.1015	−.1673	−.2377	−.3112	−.3849
.75	.0090	−.0428	−.0998	−.1615	−.2266	−.2932	−.3582
4.00	+.0054	−.0439	−.0978	−.1556	−.2159	−.2766	−.3339
.25	.0025	−.0445	−.0956	−.1499	−.2059	−.2612	−.3119
.50	.0001	−.0448	−.0933	−.1444	−.1965	−.2470	−.2920
.75	−.0019	−.0448	−.0909	−.1391	−.1877	−.2340	−.2739
5.00	−.0036	−.0446	−.0885	−.1340	−.1794	−.2221	−.2576
.25	−.0050	−.0443	−.0861	−.1292	−.1718	−.2111	−.2428
.50	−.0061	−.0438	−.0838	−.1246	−.1646	−.2010	−.2295
.75	−.0071	−.0433	−.0815	−.1203	−.1579	−.1916	−.2173
6.00	−.0079	−.0427	−.0793	−.1162	−.1517	−.1830	−.2062
.25	−.0085	−.0421	−.0771	−.1123	−.1459	−.1751	−.1962
.50	−.0091	−.0414	−.0750	−.1087	−.1404	−.1678	−.1870
.75	−.0095	−.0407	−.0731	−.1052	−.1353	−.1610	−.1785
7.00	−.0099	−.0400	−.0711	−.1019	−.1306	−.1546	−.1708

* See Sec. 6 for a description of the procedure employed.

Imaginary part of $E(z)$

r \ k	1	2	3	4	5	6	7
1.00	−.6999	−.7765	−.8510	−.9230	−.9925	−1.0593	−1.1239
.25	−.6109	−.6723	−.7294	−.7811	−.8263	−.8633	−.8906
.50	−.5411	−.5908	−.6345	−.6705	−.6969	−.7110	−.7092
.75	−.4849	−.5253	−.5586	−.5826	−.5946	−.5913	−.5675
2.00	−.4386	−.4718	−.4968	−.5115	−.5127	−.4963	−.4565
.25	−.3999	−.4272	−.4458	−.4532	−.4463	−.4204	−.3691
.50	−.3672	−.3897	−.4031	−.4050	−.3919	−.3592	−.3002
.75	−.3391	−.3577	−.3670	−.3645	−.3469	−.3094	−.2456
3.00	−.3148	−.3301	−.3361	−.3303	−.3094	−.2688	−.2022
.25	−.2935	−.3062	−.3095	−.3011	−.2778	−.2353	−.1676
.50	−.2748	−.2853	−.2864	−.2761	−.2511	−.2075	−.1399
.75	−.2582	−.2668	−.2662	−.2543	−.2283	−.1843	−.1177
4.00	−.2435	−.2504	−.2484	−.2354	−.2087	−.1649	−.0998
.25	−.2302	−.2358	−.2327	−.2188	−.1917	−.1485	−.0853
.50	−.2183	−.2227	−.2187	−.2042	−.1770	−.1345	−.0735
.75	−.2074	−.2109	−.2061	−.1912	−.1641	−.1225	−.0639
5.00	−.1976	−.2003	−.1948	−.1796	−.1528	−.1123	−.0559
.25	−.1886	−.1906	−.1846	−.1693	−.1427	−.1034	−.0494
.50	−.1803	−.1817	−.1753	−.1599	−.1338	−.0956	−.0440
.75	−.1728	−.1736	−.1669	−.1515	−.1259	−.0889	.0395
6.00	−.1658	−.1661	−.1592	−.1439	−.1188	−.0829	−.0357
.25	−.1593	−.1593	−.1521	−.1369	−.1124	−.0777	−.0325
.50	−.1534	−.1529	−.1457	−.1306	−.1066	−.0730	−.0297
.75	−.1478	−.1471	−.1397	−.1247	−.1013	−.0688	−.0274
7.00	−.1426	−.1416	−.1341	−.1194	−.0965	−.0651	−.0254

Table of $e^{i\beta_k}$

k	Real	Imaginary
1	−.20791	+.97815
2	−.40674	+.91355
3	−.58779	+.80902
4	−.74315	+.66913
5	−.86603	+.50000
6	−.95106	+.30902
7	−.99452	+.10453

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ON BENDING OF ELASTIC PLATES*

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1. Introduction. In two earlier publications[1,2] the author has considered the theory of bending of thin elastic plates with reference to the question of the boundary conditions which may be prescribed along the edges of a plate. The principal result of this work was a new system of differential equations for the deformations and stresses in thin plates. With this system of equations it is possible and necessary to satisfy three boundary conditions along the edges of a plate instead of the two conditions which Kirchhoff has first established for the classical theory.

The physical basis of these results was recognition of the fact that omission of the strain energy of the transverse shears is responsible for the contraction of the three physical boundary conditions into two conditions,** and that the problem can be treated without this omission.

While the subject is of interest from the point of view of the general theory of elasticity,[3,4] it is also of some practical importance, in particular with regard to the problem of stress concentration at the edge of holes in transversely bent plates. For such problems the classical theory leads to results which are not in accordance with experiment as soon as the diameter of the hole becomes so small as to be of the order of magnitude of the plate thickness,[5,6] while the new equations which take transverse shear deformation into account lead to results which are substantially in agreement with experiment.[7]

The main purpose of the present paper is to give an account of the author's earlier derivations[2] in simpler and more general form. While previously an isotropic homogeneous material was assumed, plates of homogeneous or non-homogeneous construction are now considered, with elastic properties which in the direction perpendicular to the plane of the plate are different from the elastic properties in directions parallel to the plane of the plate.

As a further example of application of the present system of equations, we treat the bending of a cantilever plate due to a terminal transverse load. For the homogeneous plate our result represents a minimum energy approximation to St. Venant's

* Received Aug. 7, 1946.

[1] E. Reissner, J. Math. Phys. **23**, 184–191 (1944).

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** At a free edge the three physical conditions are those of vanishing transverse force, vanishing bending couple and vanishing twisting couple. The two Kirchhoff conditions which take their places are vanishing bending couple and vanishing of the sum of transverse force and edgewise rate of change of twisting couple.

[3] A. E. H. Love, *A treatise on the mathematical theory of elasticity*, 4th ed., Cambridge University Press, Cambridge, 1927, pp. 27–29.

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[7] D. C. Drucker, J. Appl. Mech. **13**, A250–A251 (1946).

solution, while for the non-homogeneous (sandwich) plate the problem appears not to have been discussed previously.

As before, the results are obtained by an application of the basic minimum principle for the stresses and the Lagrangian multiplier method is used to obtain approximate stress strain relations. The discussion of the significance of the Lagrange multipliers is made more precise compared with that given in the earlier work, in accordance with comments which have been made.[8]

2. Statics and strain energy of plates. Let M_x and M_y be the bending couples H the twisting couple and V_x and V_y the transverse shear-stress resultants. Let p be the surface load per unit of area (Fig. 1). The equilibrium conditions for an element $dxdy$ of the plate are then

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+p=0,\qquad \frac{\partial M_x}{\partial x}+\frac{\partial H}{\partial y}-V_x=0,\qquad \frac{\partial H}{\partial x}+\frac{\partial M_y}{\partial y}-V_y=0. \tag{1}$$

Equations (1) hold regardless of the way in which the stresses are distributed over the thickness of the plate. In terms of the stresses,

$$\begin{gathered} M_x=\int_{-h/2}^{h/2} z\sigma_x dz,\qquad M_y=\int_{-h/2}^{h/2} z\sigma_y dz,\qquad H=\int_{-h/2}^{h/2} z\tau_{xy} dz, \\ V_x=\int_{-h/2}^{h/2} \tau_{xz} dz,\qquad V_y=\int_{-h/2}^{h/2} \tau_{yz} dz. \end{gathered} \tag{2}$$

Equations (1) are three equations for five unknowns. To obtain further equa-

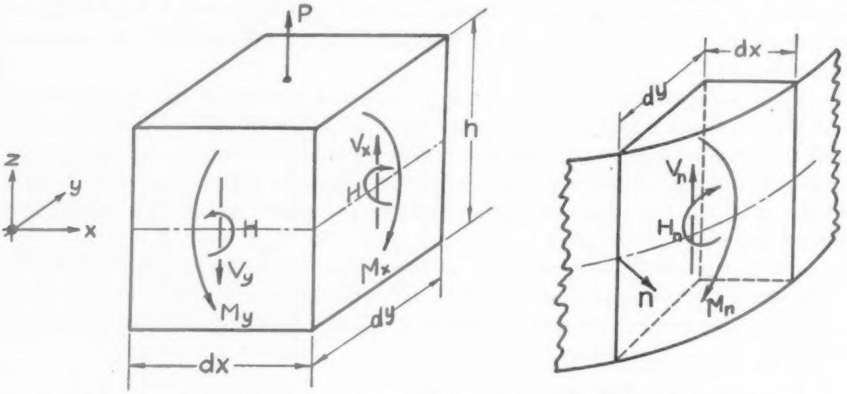

Fig. 1. Infinitesimal elements of a plate in interior and at boundary, showing orientation of stress resultants and couples.

tions, use has to be made of the stress strain relations. This is done here through the means of the basic minimum principle for the stresses (Castigliano's theorem of least work) according to which *the true state of stress is distinguished from all statically correct states of stress by the condition that the complementary energy be a minimum.*[9]

[8] J. N. Goodier, J. Appl. Mech. **13,** A251–A252 (1946).

[9] E. Trefftz in *Handbuch der Physik,* J. Springer, Berlin, 1927, vol. 6, p. 73 and Z. angew. Math. Mech. **15,** 101–108 (1935); I. S. Sokolnikoff and R. D. Specht, *Mathematical theory of elasticity,* McGraw-Hill Book Co., Inc., New York, 1946, pp. 284–287.

For a material obeying Hooke's law, and for given surface stresses or displacements, the complementary energy is the difference of the strain energy Π_s and of the work Π_b, which the surface stresses do over that portion of the surface where the displacements are prescribed.

Appropriate expressions for Π_s and Π_b are

$$\Pi_s = \frac{1}{2}\iint\left\{\frac{1}{(1-\nu^2)D}\left[M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)H\right] - \frac{2}{C_n}p(M_x+M_y) + \frac{1}{C_s}(V_x^2+V_y^2)\right\}dxdy, \tag{3}$$

$$\Pi_b = \oint (M_n\bar{\beta}_n + H_s\bar{\beta}_s + V_n\bar{w})ds. \tag{4}$$

The values of the constants D, C_n and C_s depend on the properties of the material and on the nature of the stress distribution across the thickness of the plate. Examples of their calculation for homogeneous and non-homogeneous plates will be given later on.

The functions $\bar{\beta}_n$, $\bar{\beta}_s$ and $\bar{w}$ are the generalized boundary displacements of the problem. As Π_b measures the work of the boundary stresses it follows that $\bar{\beta}_n$ must be considered as the angle through which the moment M_n turns. A corresponding definition holds for $\bar{\beta}_s$. For the same reason the quantity $\bar{w}$ is to be considered as the appropriate measure of the transverse deflection of the plate. The precise meaning of $\bar{\beta}_n$, $\bar{\beta}_s$ and $\bar{w}$, in terms of weighted averages of the three components of boundary displacement $\overline{U}_n$, $\overline{U}_s$ and $\overline{W}$, will be obtained in the following by equating the work of the boundary stresses as given by Eq. (4) to the work of the boundary stresses according to the three-dimensional theory and by reducing the expression of the three-dimensional theory to Eq. (4) by introducing the assumed variation of the stresses over the thickness of the plate.

3. Variational derivation of the stress strain relations. To make the complementary energy $\Pi_s - \Pi_b$ a minimum subject to the equations of equilibrium (1), these equations are multiplied by Lagrangian multipliers λ_a, λ_b and λ_c, respectively, and integrated over the plate area. The result is added to $\Pi_s - \Pi_b$ and the variation of the resulting expression is made to vanish:

$$\begin{aligned}\iint\Bigg\{&\frac{M_x-\nu M_y}{(1-\nu^2)D}\delta M_x + \frac{M_y-\nu M_x}{(1-\nu^2)D}\delta M_y + \frac{2(1+\nu)H}{(1-\nu^2)D}\delta H \\ &- \frac{p}{C_n}(\delta M_x+\delta M_y) + \frac{1}{C_s}(V_x\delta V_x + V_y\delta V_y) + \lambda_a\delta\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+p\right) \\ &+ \lambda_b\delta\left(\frac{\partial M_x}{\partial x}+\frac{\partial H}{\partial y}-V_x\right) + \lambda_c\delta\left(\frac{\partial H}{\partial x}+\frac{\partial M_y}{\partial y}-V_y\right)\Bigg\}dxdy \\ &+ \oint\{\bar{\beta}_n\delta M_n + \bar{\beta}_s\delta H_s + \bar{w}\delta V_n\}ds = 0.\end{aligned} \tag{5}$$

Applying Eq. (5) to a rectangular plate and eliminating variations of derivatives by integration by parts, we find that the boundary values of the Lagrangian multipliers must be

$$\bar{\lambda}_a = \bar{w}, \qquad \bar{\lambda}_b = \bar{\beta}_x, \qquad \bar{\lambda}_c = \bar{\beta}_y.$$

As Eq. (5) also holds for *any part* of the plate if the boundary displacements referring to this part are identified with the displacements occurring in the actual solution of the problem, it follows that the Lagrangian multipliers throughout the plate are related to the generalized displacements in the interior of the plate through the equations

$$\lambda_a = w, \qquad \lambda_b = \beta_x, \qquad \lambda_c = \beta_y. \tag{6}$$

Introducing Eqs. (6) into (5) and integrating by parts, we obtain the variational equation

$$\iint \left\{ \left[\frac{M_x - \nu M_y}{(1-\nu^2)D} - \frac{p}{C_n} - \frac{\partial \beta_x}{\partial x}\right]\delta M_x + \left[\frac{M_y - \nu M_x}{(1-\nu^2)D} - \frac{p}{C_n} - \frac{\partial \beta_y}{\partial y}\right]\delta M_y \right.$$
$$+ \left[\frac{2(1+\nu)H}{(1-\nu^2)D} - \frac{\partial \beta_x}{\partial y} - \frac{\partial \beta_y}{\partial x}\right]\delta H + \left[\frac{V_x}{C_s} - \frac{\partial w}{\partial x} - \beta_x\right]\delta V_x$$
$$\left. + \left[\frac{V_y}{C_s} - \frac{\partial w}{\partial y} - \beta_y\right]\delta V_y \right\} dxdy = 0. \tag{7}$$

From (7) follow the generalized stress strain relations of the problem:

$$M_x = D\left(\frac{\partial \beta_x}{\partial x} + \nu \frac{\partial \beta_y}{\partial y} + \frac{1+\nu}{C_n} p\right), \qquad M_y = D\left(\frac{\partial \beta_y}{\partial y} + \nu \frac{\partial \beta_x}{\partial x} + \frac{1+\nu}{C_n} p\right),$$
$$H = \frac{1-\nu}{2} D\left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right), \qquad \beta_x = -\frac{\partial w}{\partial x} + \frac{V_x}{C_s}, \qquad \beta_y = -\frac{\partial w}{\partial y} + \frac{V_y}{C_s}. \tag{8}$$

The conditions along a boundary $f_b(x, y) = 0$ are

$$\beta_n = \bar{\beta}_n \text{ or } M_n = \overline{M}_n, \quad \beta_s = \bar{\beta}_s \text{ or } H_s = \overline{H}_s, \quad w = \bar{w} \text{ or } V_n = \overline{V}_n. \tag{9}$$

Equations (9) are the *three* boundary conditions appropriate to the present theory when displacements or stresses are prescribed. They include the case of a free edge ($\overline{M}_n = \overline{H}_s = \overline{V}_n = 0$) and the case of a built-in edge ($\bar{\beta}_n = \bar{\beta}_s = \bar{w} = 0$). Appropriate conditions for more general edge conditions (such as elastic support) may be derived in a similar way.

The five Eqs. (8) together with the three Eqs. (1) represent a complete system of equations for the eight functions V_x, V_y, M_x, M_y, H, β_x, β_y, w. When $C_s = C_n = \infty$ they reduce to the customary equations of plate theory. To obtain the appropriate (Kirchhoff) form of the boundary conditions in this limiting case one must, however, go back to Eq. (3) and therein make $C_s = \infty$ before carrying out the remaining analysis.

4. Integration of the system of plate equations. It is possible to transform the system of Eqs. (1) and (8) such that integration in terms of harmonic and "wave" functions is possible.

The first of the equations in final form is

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = - p. \tag{10a}$$

Two further equations for V_x, V_y and w are obtained by introducing the first three Eqs. (8) into the last two Eqs. (1) and by observing the remaining Eqs. (8) and (1). The result is

$$V_x - \frac{(1-\nu)D}{2C_s}\nabla^2 V_x = - D\frac{\partial \nabla^2 w}{\partial x} - (1+\nu)D\left(\frac{1}{2C_s} - \frac{1}{C_n}\right)\frac{\partial p}{\partial x}, \tag{10b}$$

$$V_y - \frac{(1-\nu)D}{2C_s}\nabla^2 V_y = - D\frac{\partial \nabla^2 w}{\partial y} - (1+\nu)D\left(\frac{1}{2C_s} - \frac{1}{C_n}\right)\frac{\partial p}{\partial y}, \tag{10c}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Once Eqs. (10a) to (10c) are solved, the remaining five quantities M_x, M_y, H, β_x, β_y are found from Eqs. (8) by differentiations only. The first three Eqs. (8) may be written in the alternate form

$$M_x = - D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) + (1-\nu)\frac{D}{C_s}\frac{\partial V_x}{\partial x} + D\left(\frac{1+\nu}{C_n} - \frac{\nu}{C_s}\right)p, \tag{10d}$$

$$M_y = - D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) + (1-\nu)\frac{D}{C_s}\frac{\partial V_y}{\partial y} + D\left(\frac{1+\nu}{C_n} - \frac{\nu}{C_s}\right)p, \tag{10e}$$

$$H = - (1-\nu)D\frac{\partial^2 w}{\partial x \partial y} + \frac{1-\nu}{2}\frac{D}{C_s}\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right), \tag{10f}$$

where β_x and β_y have been taken from the last two Eqs. (8) and use has been made of (10a).

The system (10a) to (10f) is completed by the last two Eqs. (8).

The solution of equations (10a) to (10c) requires finding a particular integral for the load function p and finding sufficiently general solutions of the homogeneous equations. The latter is accomplished, as in the paper quoted in Footnote 2, by satisfying the homogeneous equation (10a) by means of a stress function χ in terms of which

$$V_x = \frac{\partial \chi}{\partial y}, \qquad V_y = - \frac{\partial \chi}{\partial x}\cdot \tag{11}$$

With

$$\frac{1-\nu}{2}\frac{D}{C_s} = k^2 \tag{12}$$

the homogeneous equations (10b) and (10c) become

$$\frac{\partial}{\partial y}(\chi - k^2\nabla^2\chi) = - \frac{\partial}{\partial x}(D\nabla^2 w), \qquad \frac{\partial}{\partial x}(\chi - k^2\nabla^2\chi) = \frac{\partial}{\partial y}(D\nabla^2 w). \tag{13}$$

Since Eqs. (13) are Cauchy-Riemann equations, we have

$$D\nabla^2 w - i(\chi - k^2\nabla^2\chi) = \phi + i\psi = f(x+iy). \tag{14}$$

From $\chi - k^2\nabla^2\chi = -\psi$ follows

$$\chi = \psi_1 - \psi \tag{15}$$

where ψ_1 is the general solution of the "wave" equation (with imaginary velocity of propagation)

$$\psi_1 - k^2\nabla^2\psi_1 = 0. \tag{16}$$

Thus, the stress function χ is a combination of a harmonic function ψ and a wave function ψ_1. And if the harmonic contribution to χ is taken as the imaginary part of a complex function $f(x+iy)$ then $D\nabla^2 w$ is the corresponding real part. From

$$D\nabla^2 w = \phi \tag{17}$$

it follows that, when $p=0$, w itself is a biharmonic function, just as in the theory without transverse shear deformation.

Some applications of these results to the solution of specific problems for isotropic homogeneous plates are to be found in an earlier paper.[2]

5. Homogeneous plates. The values of the constants D, C_n and C_s in the strain energy expression depend on the nature of the plate material. Their determination will now be carried out under the assumption that the material of the plate is subject to the following system of stress strain relations

$$\epsilon_x = \frac{\partial U}{\partial x} = \frac{1}{E}(\sigma_x - \nu\sigma_y) - \frac{\nu_z}{E_z}\sigma_z. \tag{18a}$$

$$\epsilon_y = \frac{\partial V}{\partial y} = \frac{1}{E}(\sigma_y - \nu\sigma_x) - \frac{\nu_z}{E_z}\sigma_z, \tag{18b}$$

$$\gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = \frac{2(1+\nu)}{E}\tau_{xy}, \tag{18c}$$

$$\epsilon_z = \frac{\partial W}{\partial z} = \frac{1}{E_z}[\sigma_z - \nu_z(\sigma_x + \sigma_y)], \tag{18d}$$

$$\gamma_{xz} = \frac{\partial W}{\partial x} + \frac{\partial U}{\partial z} = \frac{1}{G_z}\tau_{xz}, \tag{18e}$$

$$\gamma_{yz} = \frac{\partial W}{\partial y} + \frac{\partial V}{\partial z} = \frac{1}{G_z}\tau_{yz}. \tag{18f}$$

Equations (18) stipulate that the plate is isotropic with respect to directions parallel to the plane of the plate but has elastic properties in the direction normal to the plane of the plate which are different.

The strain energy for a plate of thickness h with the stress strain relations (18) is given by

$$\Pi_s = \frac{1}{2}\iint\int_{-h/2}^{h/2}\left\{\frac{1}{E}[\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y + 2(1+\nu)\tau_{xy}^2] + \frac{1}{E_z}[\sigma_z^2 - 2\nu_z\sigma_z(\sigma_x + \sigma_y)] + \frac{1}{G_z}[\tau_{xz}^2 + \tau_{yz}^2]\right\} dz\,dx\,dy. \tag{19}$$

Equation (19) is reduced to (3) by appropriate assumptions regarding the variation of the stresses across the thickness of the plate. It is rational to assume that the bending stresses vary linearly over the thickness of the plate, while the transverse shear stresses vary parabolically:

$$\sigma_x = \frac{M_x}{h^2/6}\frac{z}{h/2}, \qquad \sigma_y = \frac{M_y}{h^2/6}\frac{z}{h/2}, \qquad \tau_{xy} = \frac{H}{h^2/6}\frac{z}{h/2} \tag{20}$$

$$\tau_{xz} = \frac{V_x}{2h/3}\left[1 - \left(\frac{z}{h/2}\right)^2\right], \qquad \tau_{yz} = \frac{V_y}{2h/3}\left[1 - \left(\frac{z}{h/2}\right)^2\right]. \tag{21}$$

Equations (20) and (21) satisfy two of the three three-dimensional differential equations of equilibrium provided the stress couples and stress resultants satisfy (1). From the third of the three-dimensional equilibrium equations, and from the condition that the load p is acting on the face $z = +h/2$, the transverse normal stress σ_z is obtained:

$$\sigma_z = \frac{3p}{4}\left[\frac{2}{3} + \frac{z}{h/2} - \frac{1}{3}\left(\frac{z}{h/2}\right)^3\right]. \tag{22}$$

Substituting Eqs. (20) to (22) into Eq. (19) for Π_s, we find that the integration with respect to z may be carried out and that (19) reduces to (3).* The values of the constants D, C_n and C_s are found to be

$$D = \frac{Eh^3}{12(1-\nu^2)}, \qquad C_n = \frac{5}{6}\frac{E_z h}{\nu_z}, \qquad C_s = \frac{5}{6}G_z h. \tag{23}$$

Equations (23) are introduced into Eqs. (10). There occurs in particular

$$\begin{gathered} k^2 = \frac{1-\nu}{2}\frac{D}{C_s} = \frac{1}{10}\frac{Eh^2}{2(1+\nu)G_z}, \\ D\left(\frac{1+\nu}{C_n} - \frac{\nu}{C_s}\right) = \frac{1}{10}\frac{Eh^2}{1-\nu^2}\left(\frac{\nu_z(1+\nu)}{E_z} - \frac{\nu}{G_z}\right). \end{gathered} \tag{24}$$

For an isotropic material ($E_z = E$, $\nu_z = \nu$, $G_z = E/2(1+\nu)$) the terms in (24) reduce to the values for these quantities which were first obtained in an earlier paper of the author[2] and Eqs. (10a) to (10f) reduce to Eqs. (I) to (VI) of the earlier paper.

In order to determine the significance of the generalized displacements β_x, β_y and w, we write the work of the surface stresses in the form

$$\Pi_b = \oint \int_{-h/2}^{h/2} [\sigma_n \overline{U}_n + \tau_{ns}\overline{U}_s + \tau_{nz}\overline{W}]dzds, \tag{25}$$

where $\overline{U}_n$, $\overline{U}_s$ and $\overline{W}$ are the actual displacement components of a point of the boundary. Substituting (20) and (21) into (25), we have

$$\Pi_b = \oint \int_{-h/2}^{h/2} \left\{\frac{M_n}{h^2/6}\frac{z}{h/2}\overline{U}_n + \frac{H_s}{h^2/6}\frac{z}{h/2}\overline{U}_s + \frac{V_n}{2h/3}\left[1 - \left(\frac{z}{h/2}\right)^2\right]\overline{W}\right\} dzds. \tag{26}$$

Comparison of Eqs. (26) and (4) gives

* With the exception of a term containing p^2 which disappears when the variation is carried out and which is therefore not evaluated explicitly.

$$\bar{\beta}_n = \frac{6}{h^2}\int_{-h/2}^{h/2} \bar{U}_n \frac{z}{h/2}\,dz, \qquad \bar{\beta}_s = \frac{6}{h^2}\int_{-h/2}^{h/2} \bar{U}_s \frac{z}{h/2}\,dz,$$
$$\bar{w} = \frac{3}{2h}\int_{-h/2}^{h/2} \bar{W}\left[1 - \left(\frac{z}{h/2}\right)^2\right] dz. \tag{27}$$

As Eqs. (26) and (4) hold for any portion of the plate it follows from Eqs. (27) that throughout the interior of the plate.

$$\beta_x = \frac{6}{h^2}\int_{h/2}^{h/2} U \frac{z}{h/2}\,dz, \qquad \beta_y = \frac{6}{h^2}\int_{h/2}^{h/2} V \frac{z}{h/2}\,dz,$$
$$w = \frac{3}{2h}\int_{-h/2}^{h/2} W\left[1 - \left(\frac{z}{h/2}\right)^2\right] dz. \tag{28}$$

From Eqs. (28) it is concluded that β_x and β_y represent quantities which are equivalent to but not identical with components of change of slope of the normal to the undeformed middle surface, while w is a weighted average, taken over the thickness, of the transverse displacements of the points of the plate. Thus, according to the third Eq. (28), *the present theory leads to approximate values not for the deflection of the middle surface of the plate but for a weighted average across the thickness of the deflections of all points of the plate which lie on a normal to the middle surface.*

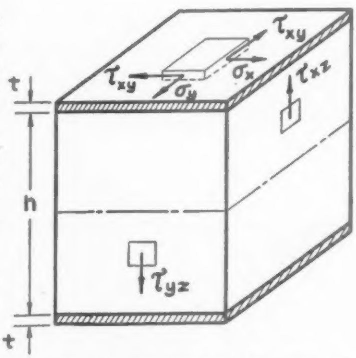

FIG. 2. Infinitesimal element of sandwich plate, showing dimensions and relevant components of stress.

6. Sandwich plates. We consider a composite plate consisting of a core layer of thickness h and of two face layers of thickness t. It is assumed that t is small compared with h and that the core material is much more flexible than the face material. Under these assumptions the transverse shears are predominantly taken by the core plate while the bending stresses are primarily taken by the face plates (Fig. 2).

We take for the strain energy of the composite plate the following expression*

$$\Pi_s = \frac{t}{E_f}\iint [\sigma_{x,f}^2 + \sigma_{y,f}^2 - 2\nu\sigma_{x,f}\sigma_{y,f} + 2(1+\nu)\tau_{xy,f}]\,dx\,dy$$
$$+ \frac{1}{2G_c}\iiint_{-h/2}^{h/2} [\tau_{xz,c}^2 + \tau_{yz,c}^2]\,dz\,dx\,dy, \tag{29}$$

where the subscript f refers to the face layers. The stresses in the face plates are taken to be uniform across the thickness and the relations between stresses and couples are then,

* While the assumptions made in what follows should give an accurate picture (within the linear theory of bending) for combinations such as a foamy core substance and aluminum face plates they will not be sufficiently accurate for plates composed for instance of two different kinds of wood.

$$\sigma_{x,f} = \pm \frac{M_x}{t(h+t)}, \qquad \sigma_{y,f} = \pm \frac{M_y}{t(h+t)}, \qquad \tau_{xy,f} = \pm \frac{H}{t(h+t)}. \tag{30}$$

As no stresses σ_x, σ_y, τ_{xy} are assumed to be acting through the core material, it follows from the differential equations of equilibrium that the transverse shear stresses do not vary across the thickness of the core,

$$\tau_{xz,c} = \frac{V_x}{h}, \qquad \tau_{yz,c} = \frac{V_y}{h}. \tag{31}$$

Substituting Eqs. (30) and (31) into (29), we obtain

$$\Pi_s = \iint \left\{ \frac{1}{t(h+t)^2 E_f} [M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)H^2] + \frac{1}{2G_c h}[V_x^2 + V_y^2] \right\} dx dy. \tag{32}$$

Comparison of equation (32) with equation (4) shows that for the sandwich plate the constants occurring in the system of differential equations (10) are given in terms of the dimensions and elastic properties of the plate as follows

$$D = \frac{1}{2}\frac{(h+t)^2 E_f}{(1-\nu^2)}, \qquad C_n = \infty, \qquad C_s = hG_c; \tag{33}$$

$$k^2 = \frac{1-\nu}{2}\frac{D}{C_s} = \frac{t(h+t)^2 E_f}{4h(1+\nu)G_c}, \quad D\left(\frac{\nu}{C_s} - \frac{1+\nu}{C_n}\right) = \frac{(\nu t(h+t)^2 E_f}{2h(1-\nu)(1+\nu)G_c} \tag{34}$$

The magnitude of the effect of transverse shear deformation is primarily determined by the magnitude of the quantity k. Comparing the first Eq. (24) for the isotropic homogeneous plate with the first Eq. (34) for the sandwich plate it is seen that the effect is of greater importance for the sandwich plate than for the isotropic plate whenever

$$\frac{th}{2}\frac{E_f}{G_c} > \frac{2(1+\nu)}{5} h^2,$$

or whenever the ratio E_f/G_c is greater than the ratio h/t.

The significance of the Lagrangian multipliers β_x, β_y and w in the present case is determined in the same manner as for the homogeneous plate. One finds here, instead of Eqs. (28), that in terms of the components of displacement U, V, W,

$$\beta_x = \frac{1}{h}\left[U\left(\frac{h}{2}\right) - U\left(-\frac{h}{2}\right)\right], \qquad \beta_y = \frac{1}{h}\left[V\left(\frac{h}{2}\right) - V\left(-\frac{h}{2}\right)\right],$$
$$w = \frac{1}{h}\int_{-h/2}^{h/2} W dz. \tag{35}$$

7. Plate equations in polar coordinates. For the applications to stress concentration problems it is convenient to have Eqs. (10) in terms of plane polar coordinates r, θ. Appropriate transformation leads to

$$\frac{\partial r V_r}{\partial r} + \frac{\partial V_\theta}{\partial \theta} = - rp, \tag{36a}$$

$$V_r - k^2\left[\nabla^2 V_r - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} - \frac{1}{r^2}V_r\right] = - D\frac{\partial \nabla^2 w}{\partial r} - (1 + \nu)D\left(\frac{1}{2C_s} - \frac{1}{C_n}\right)\frac{\partial p}{\partial r}, \tag{36b}$$

$$V_\theta - k^2\left[\nabla^2 V_\theta + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} - \frac{1}{r^2}V_\theta\right]$$

$$= - D\frac{1}{r}\frac{\partial \nabla^2 w}{\partial \theta} - (1 + \nu)D\left(\frac{1}{2C_s} - \frac{1}{C_n}\right)\frac{1}{r}\frac{\partial p}{\partial \theta}, \tag{36c}$$

$$M_r = - D\left[\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right] + 2k^2\frac{\partial V_r}{\partial r} + D\left(\frac{1 + \nu}{C_n} - \frac{\nu}{C_s}\right)p, \tag{36d}$$

$$M_\theta = - D\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial r^2}\right]$$

$$+ 2k^2\left[\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r}\right] + D\left(\frac{1 + \nu}{C_n} - \frac{\nu}{C_s}\right)p, \tag{36e}$$

$$H_{r\theta} = - (1 - \nu)D\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) + k^2\left[\frac{1}{r}\frac{\partial V_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)\right] \tag{36f}$$

$$\beta_r = - \frac{\partial w}{\partial r} + \frac{V_r}{C_s}, \quad (36g) \qquad \beta_\theta = - \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{V_\theta}{C_s}\cdot \quad (36h)$$

Equations (36d) to (36f) have been given in the paper quoted in Footnote 2 for the case of the isotropic homogeneous plate. Equations (36b) and (36c) have not previously been given. They are included in order to facilitate the obtaining of particular integrals of the system of equations for load functions of the form $p = \cos n\theta f(r)$.

Equations (11) which define the stress function χ for the solution of the homogeneous equations take on the form

$$V_r = \frac{1}{r}\frac{\partial \chi}{\partial \theta}, \qquad V_\theta = - \frac{\partial \chi}{\partial r}\cdot \tag{37}$$

Equations (15) and (16) remain unchanged:

$$\chi = \psi_1 - \psi \tag{15}$$

where $\psi(r, \theta)$ is a harmonic function and ψ_1 now satisfies the equation

$$\psi_1 - k^2\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\psi_1 = 0. \tag{16}$$

Also, as before

$$D\nabla^2 w = \phi, \tag{17}$$

where now

$$\phi(r, \theta) + i\psi(r, \theta) = f(re^{i\theta}). \tag{14}$$

Suitable expressions for ϕ, ψ and ψ_1 are given in Eqs. (42) to (46) of the paper quoted in Footnote 2. In the present formulation which includes nonisotropic, nonhomogeneous plates the quantity $h/\sqrt{10}$ in these equations is replaced by the quantity k defined in Eq. (12) above.

8. Bending of cantilever plate by terminal transverse load. As an example of the application of the formulas of this paper we may treat Saint Venant's problem of flexure of a beam with rectangular cross section.[10,11,12] Taking a plate of width $2a$ and length l, held at $x=0$, acted upon by a force P at $x=l$ and free of stress along the edges $y=\pm a$, Saint Venant's semi-inverse procedure amounts to setting

$$M_y = V_y = p = 0. \tag{38}$$

From (10a),

$$\frac{\partial V_x}{\partial x} = 0, \qquad V_x = V_x(y), \tag{39}$$

and from (10c),

$$\frac{\partial D\nabla^2 w}{\partial y} = 0, \qquad D\nabla^2 w = f(x). \tag{40}$$

Introducing (39) and (40) into (10b), we obtain

$$V_x - k^2 \frac{d^2 V_x}{dy^2} = -\frac{df}{dx}\cdot \tag{41}$$

From Eq. (41) it follows, in view of (39) and (40), that

$$V_x = C + A\cosh\frac{y}{k}, \tag{42}$$

$$D\nabla^2 w = -Cx + B. \tag{43}$$

From (43) follows

$$Dw = -C\frac{x^3}{6} + B\frac{x^2}{2} + \phi(x, y), \tag{44}$$

where $\phi(x, y)$ is a harmonic function which, according to (38) and (10e), is determined from the relation

$$M_y = -\left[\frac{\partial^2\phi}{\partial y^2} + \nu\left(\frac{\partial^2\phi}{\partial x^2} - Cx + B\right)\right] = 0. \tag{45}$$

Evaluation of Eq. (45) leads to the relation

$$Dw = \frac{-1}{1-\nu}\left(C\frac{x^3}{6} - B\frac{x^2}{2}\right) + \frac{\nu}{1-\nu}\frac{y^2}{2}(Cx - B) + Fx + I. \tag{46}$$

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[10] S. Timoshenko, Proc. London Math. Soc. (2) **20,** 398–407 (1922).

[11] S. Timoshenko, *Theory of elasticity*, McGraw-Hill Book Co., Inc., New York, 1934, pp. 292–298.

[12] A. E. H. Love, *loc. cit.*, pp. 327–346.

From Eqs. (42) and (46) follows for the relevant stress couples as defined by equations (10e) and (10f),

$$M_x = (1+\nu)(Cx - B), \qquad (47) \qquad\qquad H = -\nu C y + Ak \sinh (y/k). \qquad (48)$$

The five constants of integration A, B, C, F, I are determined by the following five conditions

$$w = \beta_x = 0 \text{ for } x = 0,\ y = 0 \tag{49}$$

$$M_x = 0, \qquad \int_{-a}^{a} V_x dy = P \quad \text{for} \quad x = l, \tag{50}$$

$$H = 0 \quad \text{for} \quad y = \pm a. \tag{51}$$

It is apparent that as in Saint Venant's theory it is not possible to satisfy the condition of complete restraint at the fixed end and also the actual distribution of the terminal load cannot be prescribed but only its resultant. As a consequence of this the solution has general validity only at distances from the ends $x=0$ and $x=l$ which are at least of the order of magnitude of the width $2a$ of the plate.

With β_x from the fourth Eq. (8), Eqs. (49) become

$$I = 0, \qquad \frac{F}{D} = \frac{C+A}{C_s}\cdot \tag{52}$$

Equations (50) become, with (47) and (42),

$$Cl - B = 0, \qquad 2aC + 2kA \sinh (a/k) = P. \tag{53}$$

Equation (51) becomes, with (48),

$$-\nu a C + Ak \sinh (a/k) = 0. \tag{54}$$

Solving Eqs. (52) to (54) and substituting into Eqs. (42), (46), (47) and (48), we obtain the following relations for the stresses and deflections

$$V_x = \frac{P}{2a}\left[1 + \frac{\nu}{1+\nu}\left(\frac{(a/k)\cosh (y/k)}{\sinh (a/k)} - 1\right)\right], \tag{55}$$

$$w = \frac{Pl^3}{2aD(1-\nu^2)}\left\{\frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{1}{6}\left(\frac{x}{l}\right)^3 - \frac{\nu}{2}\left(\frac{y}{l}\right)^2\left(1 - \frac{x}{l}\right)\right.$$
$$\left. + 2\frac{k^2}{l^2}\left(1 + \nu\frac{a/k}{\sinh (a/k)}\right)\frac{x}{l}\right\}, \tag{56}$$

$$M_x = \frac{P}{2a}\left(\frac{x}{l} - 1\right), \qquad (57) \qquad\qquad H = \frac{P}{2}\,\frac{-\nu}{1+\nu}\left[\frac{y}{a} - \frac{\sinh (y/k)}{\sinh (a/k)}\right]. \qquad (58)$$

Of particular interest is the distribution of shear stress as given by equations (55) and (58). For an isotropic plate the results are similar to a known approximate solution[10,11] for the beam with rectangular cross section. They reduce in fact to this known solution for large values of a/h.

The maximum transverse shear occurs at the ends $y = \pm a$ of the plate,

$$V_{z,\max} = \frac{P}{2a}\left[1 + \frac{\nu}{1+\nu}\left(\frac{a/k}{\tanh a/k} - 1\right)\right]. \tag{59}$$

The factor in brackets is the correction to the result of elementary beam theory. The following table gives numerical values of this factor for the isotropic plate, the orthotropic plate and the sandwich plate, in comparison with the exact values for the isotropic plate and the known approximate values for the isotropic plate.[10,11]

TABLE I. Values of Stress Concentration Factor for Transverse Shear in Cantilever Plate.

a/k	.790	1.581	3.162	6.324	9.486	12.648
$(a/h)_{\text{isotropic}}$ $(a/h)\sqrt{G_z/G}$ $(a/h)\sqrt{\dfrac{hG_c}{5tG_f}}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4
Eq. (59), $\nu_f=\frac{1}{3}$	1.050	1.180	1.545	2.331	3.121	3.91
Eq. (59), $\nu_f=\frac{1}{4}$	1.040	1.144	1.436	2.065	2.682	3.33
Appr.,[10,11] $\nu=\frac{1}{4}$	1.040	1.143	1.426	1.934		
Exact, $\nu=\frac{1}{4}$	1.033	1.126	1.396	1.988	2.582	3.176

The magnitude of the shear τ_{zy} parallel to the faces of the plate follows from equation (58). τ_{zy} is greatest at the points $(\pm h/2, \pm\eta)$ with η determined from

$$\cosh\frac{\eta}{k} = \frac{\sinh (a/k)}{a/k}\cdot \tag{60a}$$

For sufficiently large values of a/k (practically when $a/k>3$) Eq. (60a) becomes

$$\frac{\eta}{a} = 1 - \frac{\ln (a/k)}{a/k}, \tag{60b}$$

and the corresponding value of $H_{\max}$ is

$$H_{\max} = -\frac{\nu}{1+\nu}\frac{P}{2}\left[1 - \frac{\ln (a/k)+1}{a/k}\right]. \tag{61a}$$

For homogeneous plates Eq. (61a) gives for the shear stress $\tau_{zy,\max} = 6H_{\max}/h^2$,

$$\frac{2A}{3P}\tau_{zy,\max} = \frac{4}{\sqrt{10}}\sqrt{\frac{G}{G_z}}\frac{\nu}{1+\nu}\left[\frac{a}{k} - \ln\frac{a}{k} - 1\right]. \tag{61b}$$

The following table contains some values of η and of the factor in brackets in the expression for H.

TABLE II. Location and Magnitude of Maximum Shear Stress Couple and Face-Parallel Shear Stress ($\nu=.25$).

a/k	.790	1.581	3.162	6.324	12.65	∞
η/a	.578	.594	.634	.71	.80	1
$\dfrac{\eta}{a}\dfrac{\sinh (\eta/k)}{\sin (a/k)}$	.038	.129	.32	.45	.72	1
$\dfrac{2A}{3P}\sqrt{\dfrac{G_z}{G}}\tau_{zy,\max}$	.008	.0[illegible]2	.256	.72	2.30	∞
exact[13] $G_z=G$				.968	2.452	∞

By means of (59) and (58) we may calculate the ratio of maximum shear parallel to the plane of the plate and maximum transverse shear. For a homogeneous plate we have, in view of (20) and (21),

$$\frac{\tau_{xy}(\eta, h/2)}{\tau_{xz}(a, 0)} = \frac{4\nu}{1+\nu}\frac{a}{h}\left[\frac{\eta}{a} - \frac{\sinh \eta/k}{\sinh a/k}\right]\left[1 + \frac{\nu}{1+\nu}\left(\frac{a/k}{\tanh a/k} - 1\right)\right]^{-1} \tag{62a}$$

and with h/a from equation (24a)

$$\frac{\tau_{xy}(\eta, h/2)}{\tau_{xz}(a, 0)} = \frac{4}{\sqrt{10}}\sqrt{\frac{G}{G_z}}\frac{\nu}{1+\nu}\frac{a}{k}\left[\frac{\eta}{a} - \frac{\sinh \eta/k}{\sinh a/k}\right]\left[1 + \frac{\nu}{1+\nu}\left(\frac{a/k}{\tanh a/k} - 1\right)\right]^{-1} \tag{62b}$$

From equation (62b) follows in particular the limit relation

$$\lim_{a/k\to\infty} \frac{\tau_{xy}(\eta, h/2)}{\tau_{xz}(a, 0)} = \frac{4}{\sqrt{10}}\sqrt{\frac{G}{G_z}} = 1.266\sqrt{\frac{G}{G_z}} \tag{62c}$$

which is independent of Poisson's ratio. Equation (62c) shows the interesting fact that, for very thin plates, the horizontal shear may be larger than the transverse shear even for isotropic plates. We have confirmed this result for an isotropic plate by an exact calculation[13] in which the factor 1.266 is replaced by a factor 1.342.

The analogue of Eq. (62a) for sandwich plates is obtained, by means of (30), (31) and the first Eq. (34). One finds

$$\frac{\tau_{xy,f}(\eta)}{\tau_{xz,c}(a)} = \frac{\nu}{1+\nu}\sqrt{\frac{hG_f}{2tG_c}}\frac{a}{k}\left[\frac{\eta}{a} - \frac{\sinh \eta/k}{\sinh a/k}\right]\left[1 + \frac{\nu}{1+\nu}\left(\frac{a/k}{\tanh a/k} - 1\right)\right]^{-1}. \tag{63}$$

Table III contains values of the stress ratio as given by equations (62a) and (63) for a range of values of a/k and when $\nu = \frac{1}{4}$.

TABLE III. Values of Ratio of Maximum Horizontal Shear Stress to Maximum Transverse Shear Stress for Homogeneous Plates ($\nu = 1/4$) and for Sandwich Plates.

a/k	1.581	3.162	6.324	12.65	30	100	∞
$\sqrt{\frac{16tG_c}{5hG_f}}\frac{\max \tau_f}{\max \tau_c} = \sqrt{\frac{G_z}{G}}\frac{\max \tau_{xy}}{\max \tau_{xz}}$	.046	.179	.470	.695	.950	1.15	1.266

Finally, it may be indicated which form the solution assumes in plate theory without the transverse shear terms. Equations (55) and (58) become

$$V_z = \frac{P}{2a}\frac{1}{1+\nu}, \qquad H = -\frac{P}{2}\frac{y}{a}\frac{\nu}{1+\nu} \tag{64a, b}$$

Eq. (57) remains unchanged and Eq. (56) for the deflection loses the terms involving k.

The load P is thus carried *in part* by transverse shears distributed uniformly across the width of the plate and in part by means of concentrated forces at the edges $y = \pm a$ of the plate. As one would expect, no estimate is possible within the frame of the simpler theory without the transverse shear terms of the actual magnitude of the shear stresses which balance the applied load.

[13] E. Reissner and G. B. Thomas, J. Math. Phys. **25**, 241–243 (1946).

PUNCH-CARD MACHINE METHODS APPLIED TO THE SOLUTION OF THE TORSION PROBLEM*

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Introduction. Many problems in Engineering and Physics can be reduced to boundary value problems, i.e. to problems involving the determination of a function which satisfies a given partial differential equation inside a domain and assumes perscribed values on the boundary of this domain. Despite the fact that the solutions of many such problems have been known "in principle," their actual evaluation has required such a great amount of computation that it has only been possible to carry out calculations of this kind in a few simple cases.

The creation of modern computational devices such as punch-card machines, IBM-Harvard and Bell Telephone Laboratory machines, the ENIAC computer, etc., which can carry out rather extensive computations automatically, has changed the picture completely. However, these machines exist only to implement theoretical methods, and the reduction of these methods to a form whereby the machine can "take hold" represents a problem in itself.

In previous publications the author developed certain theoretical methods (the "method of orthogonal functions" and the "method of particular solutions") for solving boundary-value problems. The present paper illustrates the application of orthogonal functions to the solution of Laplace's equation $(\partial^2\phi/\partial x^2)+(\partial^2\phi/\partial y^2)=0$ through the use of punch-card machines.

1. Formulation of a problem in elasticity. The present paper is concerned with a method of solving the torsion problem for a bar of uniform cross-section.

Let x, y, Z denote rectangular coordinates, the axis of Z being perpendicular to the cross-section of the beam.[1] According to Saint Venant the components u, v, w of the displacement vector are given by the expressions

$$u = -\tau yZ, \qquad v = \tau xZ, \qquad w = \tfrac{1}{2}\tau G(x, y) \tag{1.1}$$

and the components of the stresses, X_Z and Y_Z by

$$X_Z = \mu\tau[\tfrac{1}{2}(\partial H/\partial y) - y], \qquad Y_Z = \mu\tau[x - \tfrac{1}{2}(\partial H/\partial y)]. \tag{1.2}$$

Here τ is the angle of twist per unit length, μ is the modulus of rigidity, $G(x, y)$ and $H(x, y)$ are conjugate harmonic functions.

On the boundary of the cross-section, the function $H(x, y)$ assumes the values x^2+y^2. These conditions determine G and H uniquely within an additive constant for G.

The torsion problem is thus reduced to the "first boundary value problem" of potential theory. We shall describe a method of solving it. As an illustrative example, we shall determine the function H in the case of the domain indicated in Fig. 1.

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[1] This coordinate is denoted by Z rather than z in order that this last symbol can be used to designate the complex variable $z=x+iy$.

If one of the functions, G or H, is known, the determination of the other function is very simple. Despite the fact that for the displacement components we need G in our example, we determine the function H, since its values are given on the boundary and the reader can directly estimate how much our approximate solution differs from the required values on the boundary.

2. The method of orthogonal polynomials. In this paper the method of orthogonal polynomials will be used for the determination of the function H defined above. The mathematical ideas underlying this method have been developed by the author in [1],* pp. 57–59 and supplementary Note No. II, [2] and in other papers which are listed in these references. However, the necessary computations are very involved; the aim of the present paper is to indicate how they can be performed with the use of punch-card machines.

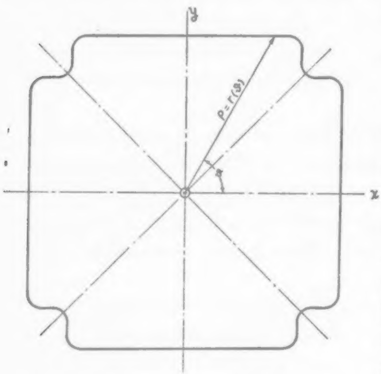

FIG. 1. The cross section B of the bar.

There exist various methods for solving the boundary-value problem of Laplace's equation. The method of orthogonal functions has the following advantages.

1) One can obtain the solution in a form in which the dependence on various parameters is evident; in particular one can obtain a solution not only for one fixed domain but for a whole family of domains which depend upon one of several parameters. For instance, in the case of the domain indicated in Fig. 1, one can obtain formulas involving the radius of curvature at the corners and thus investigate how this radius influences the stress distribution.

2) In order to solve the boundary-value problem, the bulk of the computation, i.e. the determination of orthogonal functions, need be performed only once. As soon as a set of orthogonal functions for a domain B is known the harmonic function which assumes given boundary values or given values of the normal derivative on the boundary can be easily determined.

3) The method is very general; it can be easily extended to various other problems, e.g. to problems in elasticity as well as to problems involving linear differential equations of elliptic type with non-constant coefficients (see Sec. 8).

Let $z=x+iy$, (x, y real), and let $\phi^{(n+1)}(z)$ denote the polynomial

$$\phi^{(n+1)}(z) = D_{n+1}(z)\left[E_n E_{n+1}\right]^{-1/2}, \qquad n = 0, 1, 2, \cdots \tag{2.1}$$

where D_{n+1} and E_{n+1} are the determinants

$$D_{n+1}(z) = \begin{vmatrix} F_{0,0} & F_{0,1} & \cdots & F_{0,n-1} & 1 \\ F_{1,0} & F_{1,1} & \cdots & F_{1,n-1} & z \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ F_{n,0} & F_{n,1} & \cdots & F_{n,n-1} & z^n \end{vmatrix}, \qquad E_{n+1} = \begin{vmatrix} F_{0,0} & F_{0,1} & \cdots & F_{0,n-1} & F_{0,n} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ F_{n,0} & F_{n,1} & \cdots & F_{n,n-1} & F_{n,n} \end{vmatrix} \tag{2.2a}$$

respectively, and

* Numbers in brackets refer to the bibliography at the end of this paper.

$$F_{pq} = \iint_B z^p \bar{z}^q dxdy, \qquad \bar{z} = x - iy. \tag{2.2b}$$

The set of polynomials $\{\phi^{(n)}(z)\}$, $n=1, 2, 3, \cdots$, constitutes an orthonormal set over the domain B, i.e.,

$$\iint_B \phi^{(n)}\overline{\phi^{(m)}}dxdy = \begin{cases} 1 & \text{if} \quad n = m, \\ 0 & \text{if} \quad n \neq m. \end{cases} \tag{2.3}$$

Now let

$$\Phi^{(n)}(z) = \int_0^z \phi^{(n)}(z)dz \tag{2.4}$$

and let the real and imaginary parts of $\Phi^{(n)}(z)$ be designated as ψ_n and θ_n, respectively, so that

$$\Phi^{(n)}(z) = \psi_n(x, y) + i\theta_n(x, y). \tag{2.5}$$

Note that $\Phi^{(n)}(z)$ is a polynomial of degree n in z, and that ψ_n and θ_n are polynomials of degree n in x and y.

As is well-known, both ψ_n and θ_n are harmonic functions. They possess the following property. Let $H(x, y)$ be a function which is harmonic in B and continuous on the boundary C of B. Let s denote the length along the boundary C measured from some fixed point, so that the boundary values for $H(x, y)$ may be given in the form $H(x, y)=f(s)$ on C. The function $H(x, y)$ and its conjugate $G(x, y)$ can be expressed in terms of prescribed boundary values $f(s)$ and the functions ψ_n, θ_n in the following form:

$$H(x, y) = c_1 + \sum_{\nu=1}^{\infty}\left[\psi_\nu(x, y)\int_C f(s)d\theta_\nu(s) - \theta_\nu(x, y)\int_C f(s)d\psi_\nu(s)\right], \tag{2.6a}$$

$$G(x, y) = c_2 + \sum_{\nu=1}^{\infty}\left[\theta_\nu(x, y)\int_C f(s)d\theta_\nu(s) + \psi_\nu(x, y)\int_C f(s)d\psi_\nu(s)\right], \tag{2.6b}$$

where c_k are constants. The proof of this statement may be found in [1], p. 58.

In any practical case, the infinite summation (2.6) must be approximated by a summation over a sufficiently large number of functions, i.e., (2.6a) must be replaced by[2]

$$H_N(x, y) = c_N + \sum_{\nu=1}^{N}\left[\psi_\nu(x, y)\int_C f(s)d\theta_\nu(s) - \theta_\nu(x, y)\int_C f(s)d\psi_\nu(s)\right], \tag{2.7}$$

where N is a positive integer. For increasing N the accuracy of the approximation will constantly improve. In the present case, we shall take $N=8$.

It is thus seen that the whole problem of solving the boundary-value problem is reduced to the performance of the following computations.

(i) Evaluation of the integrals

[2] In the following some quantities, e.g. θ_ν and ψ_ν, are considered as functions of different variables. In passing from one variable to another, new symbols should be introduced, since θ_ν and ψ_ν are different functions of their respective arguments. For instance

$$\theta_\nu^{(1)}(s) = \theta_\nu[x(s), y(s)].$$

For the sake of brevity the superscript (or the introduction of new symbols) will be omitted and the functions will always be denoted by θ_ν, ψ_ν, etc., irrespective of the arguments.

$$\iint_B z^p \bar{z}^q dx dy; \tag{2.8}$$

(ii) Computation of the determinants appearing in equation (2.1).

(iii) Evaluation of the coefficients

$$\int_C f(s) d\theta_\nu(s), \qquad \int_C f(s) d\psi_\nu(s); \tag{2.9}$$

(iv) Determination of the sum appearing in the right hand side of (2.7), i.e. of H_N for a sufficiently large N, and the determination of the first partial derivatives $\partial H_N/\partial x$ and $\partial H_N/\partial y$.

Using an illustrative example we shall describe in the following how each of these operations can be carried out on punch-card machines.

3. Evaluation of the integrals F_{pq}. In this paper we consider a bar with the cross-section shown in Fig. 1. This cross section possesses 8-fold symmetry.

B is a star-domain; i.e., its boundary C can be represented in polar coordinates (ρ, ϑ) in the form

$$\rho = r(\vartheta) \qquad (0 \leqq \vartheta \leqq 2\pi). \tag{3.1}$$

The values of $r(\vartheta)$, $\vartheta = 0, 1°, 2°, \cdots, 45°$ are indicated in the second column of Table 1.

TABLE 1: The values of ϑ, $r(\vartheta)$, $r^2(\vartheta)$, $H_8(r(\vartheta), \vartheta) - c_8$ and of $H_8(r(\vartheta), \vartheta)$ on the boundary C of B.

ϑ	$r(\vartheta)$	$r^2(\vartheta)$	$H_8[r(\vartheta), \vartheta] - c_8$	$H_8[r(\vartheta), \vartheta]$
0	2.722	7.41	−1.309	7.183
2	2.723	7.420	−1.295	7.197
4	2.728	7.447	−1.249	7.243
6	2.737	7.491	−1.174	7.318
8	2.749	7.557	−1.070	7.422
10	2.764	7.640	− .937	7.555
12	2.783	7.745	− .779	7.713
14	2.805	7.874	− .595	7.897
16	2.831	8.020	− .389	8.103
18	2.862	8.191	− .163	8.329
20	2.896	8.393	.078	8.570
22	2.936	8.620	.330	8.822
24	2.979	8.880	.586	9.078
26	3.028	9.169	.840	9.332
28	3.083	9.504	1.085	9.577
30	3.143	9.878	1.308	9.800
32	3.210	10.304	1.498	9.990
34	3.276	10.732	1.629	10.121
35	3.291	10.837	1.639	10.131
36	3.291	10.837	1.610	10.102
37	3.273	10.719	1.553	10.045
38	3.236	10.476	1.477	9.969
39	3.179	10.106	1.388	9.880
40	3.122	9.691	1.301	9.793
41	3.049	9.303	1.228	9.720
42	2.990	8.940	1.163	9.655
43	2.941	8.655	1.113	9.605
44	2.912	8.486	1.085	9.577
45	2.903	8.428	1.075	9.567

The double integral F_{mn} can be replaced by a single integral. Indeed,

$$
\begin{aligned}
F_{mn} &= \int_0^{2\pi}\int_0^{r(\vartheta)} \rho^{m+n+1} e^{i(n-m)\vartheta} d\rho d\vartheta \\
&= \frac{1}{m+n+2}\left\{\int_0^{2\pi} [r(\vartheta)]^{m+n+2} \cos (n-m)\vartheta d\vartheta \right. \\
&\quad \left. + i\int_0^{2\pi} [r(\vartheta)]^{m+n+2} \sin (n-m)\vartheta d\vartheta\right\}. \qquad (3.2)
\end{aligned}
$$

It follows from the aforementioned symmetry that many of the coefficients F_{mn} vanish. Indeed, $F_{mn}=0$ unless $|m-n|=0, 4, 8, \cdots$. Also, even if F_{mn} does not vanish, it can be shown that the imaginary part vanishes, so that

$$
F_{mn} = F_{nm} = \frac{1}{m+n+2}\int_0^{2\pi} [r(\vartheta)]^{m+n+2} \cos (m-n)\vartheta d\vartheta. \qquad (3.3)
$$

If we let $m+n+2=s$ and $m-n=q$, and take account of the symmetry once again, it is easily seen that

$$
F_{mn} = F_{nm} = (8/s)\int_0^{\pi/4} [r(\vartheta)]^s \cos q\vartheta d\vartheta. \qquad (3.4)
$$

Taking account of the relationship between degrees and radians, and replacing the above integral by a summation, we get the approximation:

$$
\begin{aligned}
F_{mn} = F_{nm} \approx (8/s)\cdot(\pi/180)&\left\{\tfrac{1}{2}[r(\vartheta)]^s + \tfrac{1}{2}[r(45^\circ)]^s \cos (q\pi/4)\right. \\
&\left. + \sum_{\nu=1}^{44} [r(\nu^\circ)]^s \cos (q\pi\nu^\circ/180)\right\}. \qquad (3.5)
\end{aligned}
$$

The above sum can be conveniently evaluated by using the method of "digiting without sorting," see Lorant [4, 5]. The idea underlying this procedure can be explained by using the following example. Let us assume that we have to add the following column of products:

$$
\begin{aligned}
&234 \times 24 \\
&344 \times 32 \\
&342 \times 33 \\
&232 \times 22.
\end{aligned}
$$

In the first column of the multiplicand the number 2 appears in line 1 and line 4. Instead of multiplying each number separately, we add the multipliers $(24+22=46)$ and add 46 to itself (i.e., $2\times46=46+46$). This result is accumulated. Likewise, 3 appears in the first column of the second and third multiplicands. We therefore add the corresponding multipliers $(32+33=65)$ and then multiply 65 by 3 $(3\times65=65+65+65)$. This quantity is likewise accumulated. The first column is thus completely accounted for; the same procedure is employed for the other columns. Thus, the process of multiplication, more time-consuming than addition, has been completely replaced by summations, which can be performed on an electric accounting machine.

TABLE 2: The values of F_{mn}.

m	n	F_{mn}
0	0	27.507
1	1	122.38
2	2	737.97
3	3	5,087.7
4	4	38,003.
5	5	300,110.
6	6	2,471,500.
7	7	21,041,000.
8	8	184,060,000.
4	0	−169.13
5	1	−1,523.2
6	2	−13,843.
7	3	−126,920.
8	4	−1,174,000.
8	0	−760.2

In Table 2 the values of F_{mn} obtained in our case are given[3] for $m \leqq 8$, $n \leqq 8$. It must be remembered that $F_{mn} = F_{nm}$ in this case (but not in general). All F_{nm} with $m \leqq 8$ and $n \leqq 8$ which are not listed are equal to zero.

4. Computation of the determinants. In order to determine $\phi^{(n)}(z)$, defined by Eq. (2.1), we must compute a series of determinants whose elements are complex numbers. In this section we indicate a method for evaluating determinants with complex elements by the use of punch-card machines.

We shall again explain the procedure by means of an illustrative example, referring for detailed information to Lorant [6]. Let the determinant be:

$$\begin{vmatrix} a_{11} + iA_{11} & a_{12} + iA_{12} & a_{13} + iA_{13} \\ a_{21} + iA_{21} & a_{22} + iA_{22} & a_{23} + iA_{23} \\ a_{31} + iA_{31} & a_{32} + iA_{32} & a_{33} + iA_{33} \end{vmatrix}$$

where the quantities a and A are real numbers.

We shall assume that none of the three complex numbers in the first row is equal to zero. We then take the reciprocal of each of the three numbers in the first row. The reciprocal of any complex number not equal to zero is given by the formula

$$\frac{1}{a + iA} = \frac{a}{a^2 + A^2} - \frac{iA}{a^2 + A^2}. \tag{4.1}$$

In this way we obtain the three reciprocals

$$r_k + iR_k = 1/(a_{1k} + iA_{1k}) \qquad (k = 1, 2, 3) \tag{4.2}$$

Now a card is punched for each of the nine elements of the determinant, the entries being given in Fig. 2, where n is the number of the row and m the number of the column in which the element is located. If a number is negative, we punch a hole in the corner above it. See the columns for a_{nm} and r_n, where the hole is marked by an $\times$. In this case the machines automatically replace addition by subtraction and, in multiplying, punch a hole if one of the factors has a hole.

Note that the entries in the last two columns will have indices running from 0 to 2 instead of from 1 to 3, as formerly. Now *disregard* those cards on which either or

[3] The author wishes to thank Dr. Edwin L. Crow for valuable advice and assistance in the performance of numerical computations.

1	2	3	× 4	5	× 6	7	8	9	10	11	12	13
Number of the determinant	n	m	a_{nm}	A_{nm}	r_m	R_n	$s_{nm}=a_{nm}r_m$	$S_{nm}=A_{nm}r_m$	$p_{nm}=s_{nm}-A_{nm}R_m$	$P_{nm}=S_{nm}+a_{nm}R_m$	$a^*_{n-1,m-1}=p_{nm}-p_{n1}$	$A^*_{n-1,m-1}=P_{nm}-P_{n1}$

FIG. 2. The entries on punch cards for the computation of a determinant. Every column represents a field on a punch card.

both of the two subscripts of a^* and A^* are zero; thus five of the nine cards are disregarded. From the remaining four cards we set up a new determinant

$$\begin{vmatrix} a_{11}^* + iA_{11}^* & a_{12}^* + iA_{12}^* \\ a_{21}^* + iA_{21}^* & a_{22}^* + iA_{22}^* \end{vmatrix}$$

Repetition of the above procedure reduces this determinant to a single element $a_1^{**}+iA_1^{**}$.

The value of the original determinant is then equal to

$$(a_{11} + iA_{11})(a_{12} + iA_{12})(a_{13} + iA_{13})(a_{11}^* + iA_{11}^*)(a_{12}^* + iA_{12}^*)(a_{11}^{**} + iA_{11}^{**}). \tag{4.3}$$

If we had started with a fourth-order determinant instead of a third-order one, the first stage would have reduced it to one of the third order, the second stage to one

TABLE 3: The values of a_n, b_n, etc.

n	a_n	b_n	c_n	a_n/n	$b_n/(n-4)$	$c_n/(n-8)$
1	1.9069×10^{-1}	—	—	1.9069×10^{-1}	—	—
2	9.0407×10^{-2}	—	—	4.5203×10^{-2}	—	—
3	3.6806×10^{-2}	—	—	1.2269×10^{-2}	—	—
4	1.4020×10^{-2}	—	—	3.5050×10^{-3}	—	—
5	5.2013×10^{-3}	3.1980×10^{-2}	—	1.0403×10^{-3}	3.1980×10^{-2}	—
6	1.8859×10^{-3}	2.3472×10^{-2}	—	3.1432×10^{-4}	1.1736×10^{-2}	—
7	6.7240×10^{-4}	1.2613×10^{-2}	—	9.6057×10^{-5}	4.2043×10^{-3}	—
8	2.3652×10^{-4}	5.9007×10^{-3}	—	2.9566×10^{-5}	1.4752×10^{-3}	—
9	8.2631×10^{-5}	2.6348×10^{-3}	1.8437×10^{-2}	8.1812×10^{-6}	5.2697×10^{-4}	1.8437×10^{-2}

of the second order, and a third stage would have been necessary to obtain a reduction to a single element. From this example it should be quite clear how to proceed in the general case.

Now

$$\phi^{(n)}(z) = a_n z^{n-1} + b_n z^{n-5} + c_n z^{n-9} + \cdots \tag{4.4}$$

and for ψ_n and θ_n we have the equations

$$\psi_n = (a_n/n)r^n \cos n\vartheta + (b_n/n-4)r^{n-4} \cos (n-4)\vartheta + \cdots \tag{4.5}$$

and

$$\theta_n = (a_n/n)r^n \sin n\vartheta + (b_n/n-4)r^{n-4} \sin (n-4)\vartheta + \cdots . \tag{4.6}$$

The values of a_n, b_n, c_n, a_n/n, $b_n/(n-4)$, $c_n/(n-8)$ are given in Table 3.

5. Determination of the function $H(r, \vartheta)$. In order to determine the harmonic

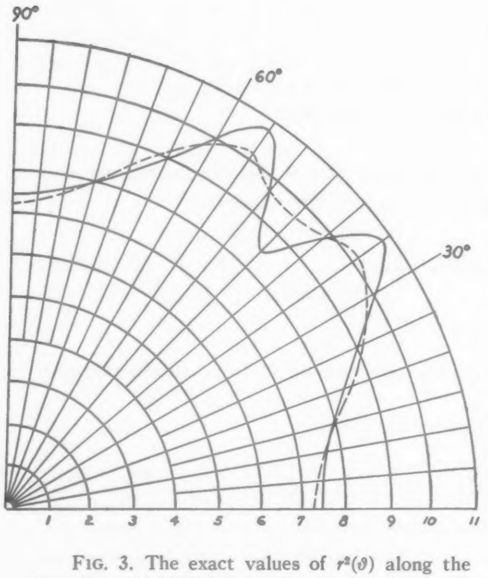

FIG. 3. The exact values of $r^2(\vartheta)$ along the boundary C of B (first quadrant) and the corresponding approximate values $H_8[r(\vartheta), \vartheta]$ indicated by dashed line.

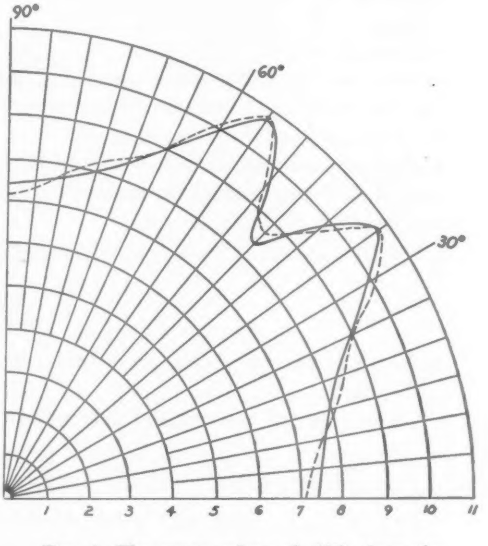

FIG. 4. The exact values of $r^2(\vartheta)$ along the boundary C of B (first quadrant) and the corresponding approximate values of $S[r(\vartheta), \vartheta]$ indicated by a dotted line.

function $H(r, \vartheta)$ which assumes the prescribed values, it remains only to determine the coefficients.

$$A_\nu = \int_C f(s)d\theta_\nu(s) \quad \text{and} \quad B_\nu = \int_C f(s)d\psi_\nu(s). \tag{5.1}$$

In our case $f(s) = x_s^2 + y_s^2 = [r(\vartheta)]^2$, where $[r(\vartheta)]^2$ is given in column 3 of Table 1. The above integral representation for A_ν can be approximated by the summation

$$A_\nu = \frac{8\pi}{180}\left\{\tfrac{1}{2}[r(0°)]^2 + \tfrac{1}{2}[r(45°)]^2 + \sum_{\mu=1}^{44}[r(\mu°)]^2[\theta_\nu(s_\mu) - \theta_\nu(s_{\mu-1})]\right\} \tag{5.2}$$

where s_μ is the point on C having the polar angle of μ degrees.

The expression for B_ν is exactly the same as (5.2) except that θ_ν is replaced by ψ_ν.

(It will be noted that instead of summing from 0° to 360° it suffices to sum from 0° to 45° and multiply by 8; this is not justified in general, but is in this particular case,

because of the symmetry of the functions θ_ν and ψ_ν and of the values of r^2 $(=x_s^2+y_s^2)$ on C.)

The summations can be easily performed by using the "method of digiting." See [4, 5]. Because of the symmetry of the specific problem under consideration, most of the coefficients vanish. In fact, it can easily be proven, and confirmed by computation, that all the A_ν will equal zero, while only B_4, B_8, etc., will differ from zero. B_4 and B_8 have been calculated and the values obtained are: $B_4=-4.734$, $B_8=-2.343$.

The function H, or more precisely the approximation H_8 obtained by taking only the first eight θ_ν and ψ_ν, is now completely determined except for the additive constant which we have designated c_8. Since the infinite sum has been replaced by a finite approximation which we may designate by H_8. it cannot be expected that r^2-H_8 will be absolutely constant. Therefore, the following process is employed in order to obtain an "average" value of c_8.

We evaluate H_8 on the boundary; i.e., we find $\sum_{\nu=1}^{8}[A_\nu\psi_\nu(s)-B_\nu\theta_\nu(s)]$ at a number of points on the boundary C. We then subtract these values from the corresponding values of r^2, and average the differences. Thus an approximate value of c_8 is obtained, and we obtain as the best possible appıoximation to $H_8(r, \vartheta)$ the quantity

TABLE 4: The values of $H_8(\rho, \vartheta)$ in B.

ϑ \ ρ	.5	1.0	1.5	2.0	2.5	3.0
0°	8.491	8.472	8.389	8.154	7.603	
5°	8.491	8.473	8.395	8.177	7.675	
10°	8.491	8.477	8.414	8.243	7.874	
15°	8.491	8.482	8.442	8.340	8.153	
20°	8.492	8.489	8.476	8.453	8.455	
25°	8.492	8.496	8.511	8.564	8.727	9.201
30°	8.493	8.502	8.544	8.661	9.936	9.531
35°	8.493	8.507	8.569	8.735	9.074	9.657
40°	8.493	8.511	8.586	8.780	9.147	8.670
45°	8.493	8.512	8.592	8.795	9.169	9.661

$$c_8+\sum_{\nu=1}^{8}[A_\nu\psi_\nu(r,\vartheta)-B_\nu\theta_\nu(r,\vartheta)]. \tag{5.3}$$

In the example under consideration $c_8=8.492$.

In the column 4 of Table 1 the values of $H_8[r(\vartheta),\vartheta]-c_8$ and in column 5 those of $H_8[r(\vartheta),\vartheta]$ are given. The values of $H_8(\rho,\vartheta)$ in B are given in Table 4.

6. Methods for improving the approximate solution. The solution obtained represents a rather rough approximation.[4] It could naturally be improved by computing more orthogonal functions and determining more terms of the development.

[4] Since the aim of the present paper is to explain the procedures used rather than to obtain the best possible numerical results, the author attempted to avoid long computations and computed only very few orthogonal functions.

In actual applications of the method one will naturally determine more of these functions, in order to obtain a much better approximation.

In many instances, however, one can save a considerable amount of labor by the use of the following simplified procedure: Suppose that there exists a domain, say M, which differs only slightly from B, see §1 and Fig. 1., and which includes $B+C$ in its interior. Let us assume further that either the function mapping M into the unit circle or a complete set of orthogonal functions $\{f_\mu(z)\}$, $\mu=1, 2, 3, \cdots$, of M are known.[5] Here we choose for M a circle of radius $R=3.295$.

Let (r_ν, ϑ_ν), $\nu=1, 2, \cdots, m$ denote the polar coordinates of m points of the boundary C of B, and q_ν the differences between the desired boundary values and those obtained by the use of orthogonal functions, i.e. let

$$q_\nu = f(r_\nu, \vartheta_\nu) - H_n(r_\nu, \vartheta_\nu), \qquad \nu = 1, 2, \cdots, m. \tag{6.1}$$

In Table 5, Columns 2, 3 and 6 the values of ϑ_ν, r_ν and q_ν, respectively are given.

TABLE 5: The values of $r(\vartheta)$, $r^2(\vartheta)$, $H_8(r_\nu, \vartheta_\nu)$, $c(r_\nu, \vartheta_\nu)$, $A(r_\nu, \vartheta_\nu)$ and $S(r_\nu, \vartheta_\nu)$ on the boundary C of B.

ν	ϑ_ν	r_ν	r_ν^2	$H_8(r_\nu, \vartheta_\nu)$	q_ν	$c(r_\nu, \vartheta_\nu)$	$A(r_\nu, \vartheta_\nu)$	$S(r_\nu, \vartheta_\nu)$
1	0	2.72	7.41	7.18	0.23	0.4	−0.02	7.16
2	10°	2.76	7.64	7.56	0.10	0.5	0.07	7.63
3	20°	2.90	8.39	8.57	−0.18	1.0	−0.09	8.48
4	25°	3.00	9.00	9.20	−0.20	2.0	−0.19	9.01
5	30°	3.14	9.88	9.80	0.08	0.9	0.32	10.12
6	35°	3.29	10.84	10.13	−0.59		0.69	10.82
7	40°	3.11	9.69	9.79	−0.10	0.2	−0.33	9.46
8	45°	2.90	8.49	9.57	−1.14	0.5	−0.98	8.59

If M is a circle of radius R we introduce the functions

$$p(R, \tau; r, \vartheta) = \frac{1}{2\pi} \cdot \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\vartheta - \tau)} \tag{6.2}$$

where are harmonic for every value of τ, and we determine l real constants T_μ, $\mu=1, 2, \cdots, l$, $l \geqq m$, so that, at the points (r_ν, ϑ_ν), the expression

$$A(r, \vartheta) = \sum_{\mu=1}^{l} T_\mu p(R, \tau_\mu; r, \vartheta) \tag{6.3}$$

equals q_ν; i.e. so that

$$\sum_{\mu=1}^{l} T_\mu p(R, \tau_\mu; r_\nu, \vartheta_\nu) = q_\nu, \qquad \nu = 1, 2, \cdots, m. \tag{6.4}$$

Remark. The conjugate $g(R, \tau; r, \vartheta)$ of $p(R, \tau; r, \vartheta)$ is given by

$$g(R, \tau; r, \vartheta) = \frac{1}{\pi} \frac{rR\sin(\vartheta - \tau)}{R^2 - 2rR\cos(\vartheta - \tau) + r^2}.$$

[5] The function $w(z, \bar{t})$ which maps M into the unit circle (taking a point t into the origin) is determined, if the set $\{f_\mu(z)\}$ is known, since according to [1, p. 53]

$$w(z, \bar{t}) = \left\{\int_0^z \left[\sum_{\mu=1}^{\infty} f_\mu(z)\overline{f_\mu(t)}\right] dz\right\}\left[\sum_{\mu=1}^{\infty} |f_\mu(t)|^2\right]^{-1/2}.$$

The determination of the quantities T_μ involves solving a system of m linear equations with n variables. This can be performed using punch card machines. The expression

$$S(r, \vartheta) = H_n(r, \vartheta) + A(r, \vartheta)$$

can be considered as a second approximation to the desired solution.

We wish to add that at the points (R, τ_μ), where the difference $R-r_\nu$ becomes very small (e.g. for $\tau_\mu = 35°$ in our example), one has to replace $p(R, \tau_\mu; r, \vartheta)$ by

$$\omega(R, \tau_\mu^{(0)}, \tau_\mu^{(1)}; r, \vartheta) = \int_{\tau_\mu(0)}^{\tau_\mu(1)} p(R, \tau; r, \vartheta)d\tau, \tag{6.5}$$

where $\tau_\mu^{(0)}$ $\tau_\mu^{(1)}$ $(\tau_\mu^{(0)} < \tau_\mu < \tau_\mu^{(1)})$ are suitably chosen quantities.

We note that the expression (6.3) can be considered as an approximation of the integral

$$\int_0^{2\pi} p(R, \tau; r, \vartheta)h(\tau)d\tau = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)h(\tau)d\tau}{R^2 + r^2 - 2rR\cos(\vartheta - \tau)}, \tag{6.6}$$

where $h(\tau)$ is some continuous function which at the points τ_μ assumes the values $[T_\mu/(\tau_\mu^{(0)} - \tau_\mu^{(1)})]$. In many instances instead of solving Eq. (6.4), we can estimate the values of T_μ, taking T_μ approximately equal to q_ν.

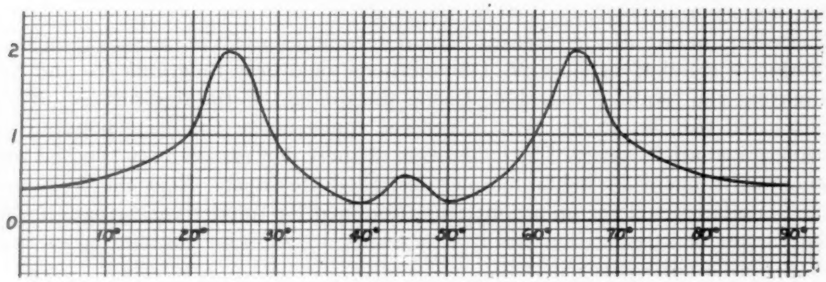

FIG. 5. The values of $\sum_{k=0}^{3}[p(R, 20°+k\cdot 90°; r(\vartheta), \vartheta)+p(R, 70°+k\cdot 90°; r(\vartheta), \vartheta)]$, considered as function of ϑ, along the boundary C of B.

In the example under consideration we took for[7] $A(r, \vartheta)$,

$$\begin{aligned} A(r, \vartheta) \quad & -0.2\sum_{k=0}^{3}[p(R, 25° + k\cdot 90°; r, \vartheta) + p(R, 65° + k\cdot 90°; r, \vartheta)] \\ & + 0.706\sum_{k=0}^{3}[p(R, 35° + k\cdot 90°; r, \vartheta) + p(R, 55° + k\cdot 90°; r, \vartheta)] \\ & - 1.139\sum_{k=0}^{3} p(R, 45° + k\cdot 90°; r, \vartheta), \end{aligned} \tag{6.7}$$

$$R = 3.295.$$

[7] We note that since $r(35°)$ is almost equal to 3.295, we make an over-simplification by replacing

$$0.706[(\tau_6^{(1)} - \tau_6^{(0)})\omega(R, \tau_6^{(1)}, \tau_6^{(0)}; r, \vartheta) + \cdots] \text{ by } 0.706\,[p(R, 35°, r, \vartheta) + \cdots];$$

this approximation, although valid approximately for points which are far enough away from $(R, 35°+k\cdot 90°)$, $(R, 55°+k\cdot 90°)$, $k=0, 1, 2, 3$, would have to be replaced by more exact expressions in the neighborhood of these points.

The values of

$$c(r, \vartheta) = \sum_{k=0}^{3} [p(R, 25° + k\cdot 90°; r, \vartheta) + p(R, 65° + k\cdot 90°; r, \vartheta)] \qquad (6.8)$$

at points (r_ν, ϑ_ν) are given in column 7 of Table 5 and drawn in Fig. 5.

The values of $A(r, \vartheta)$ and $S(r, \vartheta)$ are indicated in columns 8 and 9 of Table 5. In the case where M is not a circle, but the mapping function of M into the unit circle is known, we can proceed similarly. A very convenient procedure which can be applied in this case consists in the use of curvilinear coordinates (in G) which correspond to polar coordinates in the circle.

On the other hand, if a complete system of orthogonal functions $\{f_\mu\}$, $\mu=1, 2, \cdots$ of the domain M is known, we can construct in M a set of functions P_μ which possess properties similar to the functions $p(R, \tau_\mu; r, \vartheta)$.

The characteristic property of the function $p(R, \tau; r, \vartheta)$ is that it is harmonic in M, it becomes infinite at the point $(r, \vartheta) = (R, \tau)$ and vanishes on the remaining part of the boundary of M.

We denote by P_μ a system of functions which approximately equal unity on a small interval of the boundary and zero on the remaining part. Using the system $\{f_\nu\}$ of orthogonal functions, we can determine a set of functions P_μ, and then proceed as before, replacing the $p(R, \tau_\mu; r, \vartheta)$ by the P_μ's.

7. Determination of the stresses. The determination of the stresses requires the evaluation of the derivative of H. Our methods yield only functions H_N and S, approximating the required function. Inside the domain B the derivatives $(\partial H_N/\partial x, \partial H_N/\partial y)$ and $(\partial S/\partial x, \partial S/\partial y)$ will in general approximate the corresponding exact solution quite satisfactorily. The evaluation of these derivatives may proceed by use of punch-card machines, in the manner described previously. Near the boundary or on the boundary itself the approximations obtained for the derivatives will, in many instances, not be satisfactory, and it is advisable to use some summation method in order to obtain more exact values.

In particular at sharp edges, or points where the radius of curvature of the boundary is no longer continuous, it is necessary to apply special methods for the evaluation of the derivatives. These questions require special treatment, and will be discussed in another paper.

8. Additional remarks. It should be noted that the method described in the present paper can be extended without difficulty to the study of thin plates as well as to the general problem of elasticity in three dimensions (see [8]).

Up to the present time, it has been practically impossible to determine Green's function for the equation $\Delta u = 0$ in the case of multiply-connected domains. In a recent paper [9] Schiffer has indicated a simple relation between Green's function of Laplace's equations and systems of orthonormal functions which permits a comparatively simple determination of Green's function of such domains.

Finally by introducing some additional considerations the method of orthogonal functions can be extended to a large class of linear partial differential equations in two and three variables (see [10], [11]).

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THE REFLECTION OF AN ELECTROMAGNETIC PLANE WAVE BY AN INFINITE SET OF PLATES, II*

BY

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1. Introduction. In Part I[4] we have calculated rigorously with Fourier transform methods, the reflection and transmission coefficients due to the incidence of a plane electromagnetic wave on an infinite set of parallel staggered plates. We discussed there the case in which there was only one component of the electric field excited; that is, the incident electric field was parallel to the edges of the plates. We shall now discuss the same geometric structure when it is excited by a plane wave which has only a single component of the magnetic field which is parallel to the edges of the plates. In this case we shall see that the magnitude of the reflection and transmission coefficients are independent of the wavelength and depend only on the angle of stagger α, and the direction of incidence of the plane wave θ. We shall use the Fourier transform technique again, and since many of the calculations are parallel to those which we did in Part I, we shall only outline the procedure.

2. Formulation of the problem. We treat here the following problem. A plane monochromatic electromagnetic wave whose direction of propagation lies in the plane of the paper, is incident upon an infinite set of staggered, equally spaced, semi-infinite metallic plates of zero thickness and perfect conductivity. These plates extend indefinitely in a direction perpendicular to the plane of the paper. (See Fig. 1 for a side view.) The angle of stagger with respect to a fixed direction (that of the cross section of the plates in Fig. 1) is α, while the direction of propagation with respect to this fixed direction is θ, where $\alpha-\pi<\theta<\alpha$ and $\theta<\alpha\leqq\pi/2$.

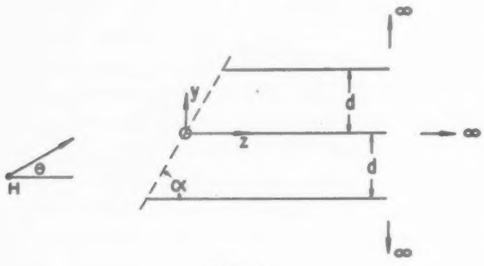

FIG. 1

We assume here that the incident wave has only one component of the magnetic field, that is, the component which is perpendicular to the plane of the paper. For such an excitation, no other components of the magnetic field can be excited. Thus, all components of the electric field can be derived from a single component of the magnetic field $H_x(y, z)=\psi(y, z)$. For this case, the components of the electric field lie in

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[3] This paper is based on work done for the Office of Scientific Research and Development under Contract OEMsr-262 with the Massachusetts Institute of Technology.

[4] J. F. Carlson and A. E. Heins, *The reflection of an electromagnetic plane wave by an infinite set of plates*, I, this Quarterly, **4**, 313–329 (1947). Hereafter we shall refer to this paper as I.

the plane of the paper and we shall refer to this problem as an "E plane" problem.

Let us assume as in I, that the time dependence of all field quantities is e^{-ikct} so that Maxwell's equations may be written in spatial form as

$$\nabla \times \mathbf{E} = ik\mathbf{H}$$

and

$$\nabla \times \mathbf{H} = -ik\mathbf{E},$$

where $k=2\pi/\lambda$, and λ is the free space wave-length. We then have that $ikE_y = -\partial\psi/\partial z$ and $ikE_z = \partial\psi/\partial y$. Upon eliminating E_y and E_z from the above equations we get the two dimensional wave equation

$$\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0$$

which is to be solved subject to the boundary condition $\partial\psi/\partial y = 0$ on the metal plates, since E_z is the tangential component of the electric field on these plates. There are also conditions at infinity on the function $\psi(y, z)$ which are similar to those which we required for $\phi(x, z)$ in I.

We now formulate the equation which expresses the z component of the electric field at any point (y, z) in terms of the surface current density on the plates. Following a procedure similar to that developed in I, we find

$$\psi(y, z) = \psi_{\text{in}}^{c}(y, z) - \frac{i}{4} \sum_{m=-\infty}^{\infty} \int_{mg}^{\infty} I_m(z') \frac{\partial H_0^{(1)}}{\partial y'} [k\sqrt{(z - z')^2 + (y + y')^2}]dz' \tag{2.1}$$

where $y' = md$, $g = d \cot \alpha$ and $\psi_{\text{inc}}(y, z) = e^{ik(y\sin\theta + z\cos\theta)}$ is the incident magnetic field. $H_0^{(1)}[k\sqrt{(x-z')^2+(y-y')^2}]$ is the free space Green's function which we have described in I. Due to the form in which y and y' appear, we may write (2.1) as

$$\psi(y, z) = \psi_{\text{inc}}(y, z) + \frac{i}{4} \frac{\partial}{\partial y} \sum_{m=-\infty}^{\infty} \int_{mg}^{\infty} I_m(z') H_0^{(1)} [k\sqrt{(z - z')^2 + (y - md)^2}]dz'.$$

Now $\partial\psi/\partial y$, which is proportional to $E_z(y, z)$, is given by the equation

$$\frac{\partial}{\partial y}\psi(y, z) = \frac{\partial}{\partial y}\psi_{\text{inc}}(y, z) + \frac{i}{4} \frac{\partial^2}{\partial y^2} \sum_{m=-\infty}^{\infty} \int_{mg} I_m(z') H_0^{(1)} [k\sqrt{(z - z')^2 + (y - md)^2}]dz', \tag{2.2}$$

and since $E_z(y, z)$ vanishes for $y = nd$, $z > nd \cot \alpha$, $n = 0, \pm 1, \cdots$ Eq. (2.2) leads to an infinite set of inhomogeneous integral equations of the Wiener-Hopf type similar to those developed in I.

Some remarks are now in order about the range of values of d/λ which we will assume here. In the first place, we suppose that in the parallel plate regions, $\psi(y, z)$ is proportional to e^{ikz} for z sufficiently large and positive. That is, the parallel plate region can sustain the so-called principal mode, a mode which cannot occur in the H plane case. In order that no higher mode propagate, we must now have that $0 < d/\lambda < \frac{1}{2}$. We further assume that there is only a single reflected wave. This assumption puts further limitations on d/λ as well as on θ, and we shall discuss them when we have obtained the final solution of the problem.

3. Fourier transform solution of the integral equation. We have already shown in I, Section 3, that the surface current density on any plate can be expressed in terms of the surface current density of a given plate, the argument being that we had a geometric structure with two types of periodicities. The same argument shows us now that

$$I_m(z - mg) = I_0(z)e^{ikm(g\cos\theta + d\sin\theta)},$$

where $I_0(z)$ is the surface current density on the zeroth plate. With this reduction and an appropriate change of variables, the integral equation which we have to solve, now reads

$$0 = \frac{\partial}{\partial y}\psi_{\text{inc}}(y, z)$$
$$+ \frac{i}{4}\sum_{m=-\infty}^{\infty}\int_0^{\infty} I_0(z')e^{ik\mu m}\frac{\partial^2}{\partial y^2}H_0^{(1)}\left[k\sqrt{\{z-z'+(n-m)g\}^2+(y-md)^2}\right]dz' \quad z>0, \quad (3.1)$$

where now $g = d\cot\alpha$, and $\mu = g\cos\theta + d\sin\theta$. The y derivatives in (3.1) are understood to be evaluated at $y = nd$. Equation (3.1) is an inhomogeneous Wiener-Hopf integral equation, which may be rewritten as

$$0 = ik\sin\theta e^{ikz\cos\theta}$$
$$+ \frac{i}{4}\sum_{p=-\infty}^{\infty}\int_0^{\infty} I_0(z')e^{ik\mu p}\frac{\partial^2}{\partial y^2}H_0^{(1)}\left[k\sqrt{(z-z'+pg)^2+(y-nd-pd)^2}\right]dz' z>0. \quad (3.2)$$

The dependence of Eq. (3.2) on n is not explicit as the differentiation of $H_0^{(1)}\left[k\sqrt{(z-z'+pg)^2+(y-nd-pd)^2}\right]$ with respect to y will demonstrate. We now extend Eq. (3.2) to hold for all z by writing it as

$$F(z) = \frac{\partial}{\partial y}\psi_{\text{inc}}(y, z)\Big|_{y=0}$$
$$+ \frac{i}{4}\sum_{p=-\infty}^{\infty}\int_{-\infty}^{\infty} I_0(z')e^{ik\mu p}\frac{\partial^2}{\partial y^2}H_0^{(1)}\left[k\sqrt{(z-z'+pg)^2+(y-nd-pd)^2}dz'\right., \quad (3.3)$$

where we define

$$F(z) \equiv 0 \quad z > 0, \qquad I_0(z) \equiv 0 \quad z < 0,$$

$$\frac{\partial}{\partial y}\psi_{\text{inc}}(y, z)\Big|_{y=0} \begin{cases} \equiv ik\sin\theta e^{ikz\cos\theta} & z > 0, \\ \equiv 0 & z < 0. \end{cases}$$

We assume as in I, for analytical convenience, that k has a small positive imaginary part.

By arguments which we employed in I, we can show that $F(z)$ is asymptotic to $e^{ikz\cos(2\alpha-\theta)}$ for z large and negative and therefore it has a Fourier transform

$$\int_{-\infty}^{0} e^{-iwz'}F(z')dz'$$

which is regular in the upper half of the ω plane $\mathfrak{Im}\, w > +\mathfrak{Im}\, k\cos(2\alpha - \theta)$. Since $I_0(z)$ is asymptotic to e^{ikz} for z large and positive, that is, the parallel plate regions are of such spacing as to permit the propagation of the principal mode only, the Fourier

transform of $I_0(z)$ is regular in the lower half of the w plane $\mathfrak{Im}\, w < \mathfrak{Im}\, k$.[5] The transform of $\partial\psi_{inc}(y, z)/\partial y|_{y=0}$ is regular in the lower half of the w plane, $\mathfrak{Im}\, w < \mathfrak{Im}\, k \cos\theta$. Finally, the transform of the kernel of Eq. (3.3) is

$$\frac{\sqrt{k^2 - w^2} \sin d\sqrt{k^2 - w^2}}{\cos d\sqrt{k^2 - w^2} - \cos(k\mu - wd)} = K(w) = \frac{K_-(w)}{K_+(w)}$$

and it is regular in the strip

$$\mathfrak{Im}\, k \cos(2\alpha - \theta) < \mathfrak{Im}\, w < \mathfrak{Im}\, k \cos\theta.$$

Since all of the transforms involved in Eq. (3.3) have a common strip of regularity in the w plane, it is thus legitimate to apply the Fourier transform to this equation to get

$$D(w) = \frac{k \sin\theta}{w - k\cos\theta} - \frac{\sqrt{k^2 - w^2} \sin d\sqrt{k^2 - w^2}\, J(w)}{2[\cos d\sqrt{k^2 - w^2} - \cos(k\mu - wd)]}, \tag{3.4}$$

where $D(w)$ and $J(w)$ are the respective transforms of $F(z)$ and $I_0(z)$.

We are once again led to a problem in factorization of the coefficient of $J(w)$ into two functions, one of which is regular in the upper half plane $\mathfrak{Im}\, w > \mathfrak{Im}\, k \cos(2\alpha - \theta)$, while the other is regular in the lower half plane $\mathfrak{Im}\, w < \mathfrak{Im}\, k \cos\theta$. The factorization of the denominator $\cos d\sqrt{k^2 - w^2} - \cos(k\mu - wd)$ is the same as it was in I, save for the fact that ρ and a have been replaced by μ and d respectively. The numerator $\sqrt{k^2 - w^2} \sin d\sqrt{k^2 - w^2}$ may be written in factor form as

$$d(k^2 - w^2) \prod_{n=1}^{\infty} [\sqrt{1 - (kd/n\pi)^2} + (iwd/n\pi)] e^{-iwd/n\pi}$$
$$\cdot \prod_{n=1}^{\infty} [\sqrt{1 - (kd/n\pi)^2} - (iwd/n\pi)] e^{iwd/n\pi}.$$

Clearly $(k - w)\prod_{n=1}^{\infty} [\sqrt{1 - (kd/nx)^2} + iwd/nx] e^{-iwd/nx}$ has no zeros in the lower half plane $\mathfrak{Im}\, w < \mathfrak{Im}\, k$, while the factor $(k + w)\prod_{n=1}^{\infty} [\sqrt{1 - (kd/nx)^2} - (iwd/nx)] e^{iwd/nx}$ has no zeros in the upper half plane $\mathfrak{Im}\, w > \mathfrak{Im}(-k)$. Thus

$$\frac{\sqrt{k^2 - w^2} \sin d\sqrt{k^2 - w^2}}{\cos d\sqrt{k^2 - w^2} - \cos(k\mu - wd)}$$

may be factored into two functions, $K_+(w)$ and $K_-(w)$, where $K_+(w)$ is regular in the upper half plane $\mathfrak{Im}\, w > \mathfrak{Im}\, k \cos(2\alpha - \theta)$, while $K_-(w)$ is regular in the lower half plane $\mathfrak{Im} < \mathfrak{Im}\, k \cos\theta$. We have

$$K_-(w) = \left[(k - w) \prod_{n=1}^{\infty} [\sqrt{1 - (kd/n\pi)^2} + (iwd/n\pi)] e^{-iwd/n\pi} e^{\chi(w)}\right]$$
$$\cdot \left[(w - \sigma_1) \prod_{n=-\infty}^{-1} [\Delta_n - i\Psi_n] e^{(k\mu - wg + iwd)/2n\pi + i(\frac{1}{2}\pi - \alpha)} \prod_{n=1}^{\infty} [\Delta_n + i\Psi_n] e^{(k\mu - wg + iwd)/2n\pi - i(\frac{1}{2}\pi - \alpha)}\right]$$

[5] It has been tacitly assumed that $F(z)$ and $I_0(z)$ are integrable for all finite z. This we can show directly from the integral equation.

and

$$\frac{1}{K_+(w)}=\frac{2d}{(d^2+g^2)}\left[(k+w)\prod_{n=1}^{\infty}\left[\sqrt{1-(kd/n\pi)^2}-(iwd/n\pi)\right]e^{iwd/n\pi}e^{-\chi(w)}\right]$$
$$\cdot\left[(w-\sigma_2)\prod_{n=-\infty}^{-1}\left[\Delta_n+i\Psi_n\right]e^{(k\mu-wg+iwd)/2n\pi-i(\frac{1}{2}\pi-\alpha)}\prod_{n=1}^{\infty}\left[\Delta_n-i\Psi_n\right]e^{(k\mu-wg+iwd)/2n\pi+i(\frac{1}{2}\pi-\alpha)}\right]$$

The Δ_n and Ψ_n are the same functions of w which we encountered in I save for the interchange of ρ and a, by μ and d respectively; $\sigma_1=k\cos\theta$, $\sigma_2=k\cos(2\alpha-\theta)$ while $\chi(w)$ is an integral function introduced into the decomposition of $K(w)$ to render the portions $K_+(w)$ and $K_-(w)$ algebraic in growth in their respective half planes of regularity as $|w|$ becomes infinite.

The decomposition of Eq. (3.4) into two functions, one of which is regular in the upper half plane $\mathfrak{Im}\,w>\mathfrak{Im}\,k\cos(2\alpha-\theta)$ and the other of which is regular in the lower half plane $\mathfrak{Im}\,w<\mathfrak{Im}\,k\cos\theta$ follows as before. We now have

$$D(w)K_+(w)-\frac{k\sin\theta[K_+(w)-K_+(k\cos\theta)]}{w-k\cos\theta}$$
$$=-\frac{K_-(w)}{2}J(w)+\frac{kK_+(k\cos\theta)\sin\theta}{w-k\cos\theta}\cdot \tag{3.5}$$

Since the left side of (3.4) is regular in the lower half plane $\mathfrak{Im}\,w<\mathfrak{Im}\,k\cos\theta$ and the right side is regular in the upper half plane $\mathfrak{Im}\,w>\mathfrak{Im}\,k\cos(2\alpha-\theta)$, while both sides are regular in the strip $\mathfrak{Im}\,k\cos(2\alpha-\theta)<\mathfrak{Im}\,w<k\cos\theta$, it follows that both sides of Eq. (3.5) are equal to an integral function. The final decomposition then gives us

$$D(w)K_+(w)-\frac{k\sin\theta[K_+(w)-K_+(k\cos\theta)]}{w-k\cos\theta}=\text{Integral function}, \tag{3.6}$$

$$-\tfrac{1}{2}J(w)K_-(w)+\frac{kK_+(k\cos\theta)\sin\theta}{w-k\cos\theta}=\text{Integral function}. \tag{3.7}$$

We now turn to the evaluation of the integral function of separation and $\chi(w)$. The functional form of $\chi(w)$ follows from the asymptotic form of $K_-(w)$, $|w|\to\infty$, $\mathfrak{Im}\,w<0$ or of $K_+(w)$, $|w|\to\infty$, $\mathfrak{Im}\,w>0$. The method of calculating these asymptotic forms is the same as in I. We find now that

$$K_-(w)\sim w^{-1/2},\qquad \mathfrak{Im}\,w<0\qquad |w|\to\infty,$$

provided $\chi(w)$ is chosen as $-(iwd/\pi)[(\alpha-\pi/2\cot\alpha)-ln2\sin\alpha]$. If we let $|w|\to\infty$, $\mathfrak{Im}\,w<0$, it is clear that the integral function of separation in Eq. (3.7) approaches zero in the lower half plane. A similar calculation for Eq. (3.6) shows that the integral function approaches zero in the upper half plane $|w|\to\infty$, $\mathfrak{Im}\,w>0$. Since this integral function is everywhere bounded and approaches zero for $|w|\to\infty$, it is identically zero. From (3.7) we then get the Fourier transform of $I_0(z)$

$$J(w)=\frac{2kK_+(k\cos\theta)\sin\theta}{K_-(w)[w-k\cos\theta]}\cdot \tag{3.8}$$

4. Investigation of the far fields. In order to find the amplitudes of the reflected

and transmitted waves we must find the asymptotic form of $\psi(y, z)$ as $|z| \to \infty$. This can be done by noting that Eq. (2.1) may be written in Fourier integral representation as

$$\psi(x, z) = \psi_{\text{inc}}(y, z) - \frac{1}{4x}\int_C dwJ(w)e^{iwz+i(k\mu-wg)n}$$
$$\cdot \frac{\cos (y - dn - d)\sqrt{k^2 - w^2} - e^{i(k\mu-wg)} \cos (dn - y)\sqrt{k^2 - w^2}}{\cos \sqrt{k^2 - w^2}\, d - \cos (k\mu - wg)} \tag{4.1}$$

where the contour C is a path in the strip of regularity $\mathfrak{Im}k \cos (2\alpha-\theta) < \mathfrak{Im}w < \mathfrak{Im}k \cos \theta$ and is closed above or below by a semi-circle which passes between the poles of the integrand, depending on whether z is greater or less than zero. The integration along the semi-circular arc gives no contribution to the integral (4.1) when its radius becomes infinite. n is the largest integer in y/d.

Let us first consider the asymptotic form of $\psi(y, z)$ as z becomes large and positive. In the integral in Eq. (4.1), we see that the two poles which give propagating mode contributions are at $w = \sigma_1 = k \cos \theta$, and $w = k$. All other poles give exponential factors in z which attenuate for z large and positive. The contour C is, of course, closed in the upper half plane. The integral in (4.1) is then

$$- \frac{1}{2\pi}\int_C dw \frac{kK_+(k \cos \theta) \sin \theta e^{iwz}e^{i(k\mu-wg)n}}{(w - k \cos \theta)\sqrt{k^2 - w^2}\, K_+(w) \sin d\sqrt{k^2 - w^2}}$$
$$\cdot \left[\cos (y - dn - d)\sqrt{k^2 - w^2} - e^{i(k\mu-wg)} \cos (dn - y)\sqrt{k^2 - w^2}\right]$$
$$= - e^{ik(z\cos\theta+y\sin\theta)}$$
$$- \frac{ik \sin \theta K_+(k \cos \theta)e^{ikz+iknd\sin\theta}}{2kd(1 - \cos \theta)K_+(k)}(1 - e^{i(k\mu-kg)})$$

+ exponential terms in y and z which are attenuated for z large and positive.

Thus, for z large and positive, $\psi(y, z)$ is asymptotic to

$$\frac{- i \sin \theta K_+(k \cos \theta)e^{iknd\sin\theta+ikz}(i - e^{ikd\sin\theta})}{2kdK_+(k)(1 - \cos \theta)},$$

that is, the correct lowest mode functional form in the parallel plate region for "E plane" polarization. The magnitude $|T|$ of the transmission coefficient is then

$$|T| = \frac{\sin \theta \left| K_+(k \cos \theta) \sin [k(\mu - g)/2]\right|}{kd(1 - \cos \theta) \left| K_+(k) \right|}, \tag{4.2}$$

which reduces to

$$|T| = \frac{\sin (\alpha - \theta)}{\sin (\alpha - \frac{1}{2}\theta) \cos \frac{1}{2}\theta}$$

provided now k is taken to be purely real.

We now compute the amplitude of the reflected wave by the same method that we employed in I. That is, for y fixed and z large and negative, we close the path by a large semi-circular arc in the lower half plane which passes between the poles in this half plane. If we allow the radius of this semi-circle to become infinite, the usual

arguments show that there is no contribution to the integral from this circular arc. The residue calculation then gives us for y fixed, z large and negative,

$$\psi(y, z) = \psi_{\text{inc}}(y, z) + \frac{e^{ik[y\sin(2\alpha-\theta)+z\cos(2\alpha-\theta)]}K_+(k\cos\theta)\sin\theta}{2kK'_+[k\cos(2\alpha-\theta)]\sin\alpha\sin(2\alpha-\theta)\sin(\alpha-\theta)}$$

$$+ \text{ terms which attenuate exponentially for } z \text{ large and negative.}$$

With the amplitude of the incident wave taken as unity, the amplitude of the reflected wave, that is, the reflection coefficient is

$$R = \frac{K_+(k\cos\theta)\sin\theta}{2kK'_+[k\cos(2\alpha-\theta)]\sin\alpha\sin(2\alpha-\theta)\sin(\alpha-\theta)}$$

which simplifies to

$$R = \tan\tfrac{1}{2}\theta\cot(\alpha-\tfrac{1}{2}\theta)e^{i\Phi}$$

when k is taken real. Φ is the phase angle of the reflection coefficient and is of the same functional form as the $\Theta_1' - \Theta_2'$ found in I, Section 5, save now for the fact that a has been replaced by d. We have finally

$$|R| = \tan\tfrac{1}{2}\theta\cot(\alpha-\tfrac{1}{2}\theta).$$

Conservation of power flow from free space to any parallel plate region gives the following relation between $|R|$ and $|T|$:

$$\frac{\sin(\alpha-\theta)}{\sin\alpha}[1-|R|^2] = |T|^2.$$

The values of $|R|$ and $|T|$ clearly satisfy this condition. Finally, we note that the condition for a single reflected wave is the same as it was in I, namely

$$\frac{2d}{\lambda} < \frac{\sin\alpha}{\cos^2[(\theta-\alpha)/2]}\,. \tag{4.3}$$

Since $0<d/\lambda<\frac{1}{2}$, the inequality (4.3) is not as severe as the comparable relation in I.

It is illuminating to introduce an angle i into the formulas (4.2) and (4.3). This angle i is the angle which the direction of propagation of the incident wave makes with the normal to the trace of the edges of the parallel plates and is equal to $\frac{1}{2}\pi-\alpha+\theta$. The magnitude of the reflection coefficient then becomes

$$|R| = \frac{\sin\alpha - \cos i}{\sin\alpha + \cos i},$$

while the magnitude of the transmission coefficient becomes

$$|T| = \frac{2\cos i}{\sin\alpha + \cos i}\,.$$

There will be no reflection from this structure if $|R|=0$ or $\alpha=\frac{1}{2}\pi-i$, that is, the direction of propagation of the incident wave is parallel to surfaces of the parallel plates. In this case, of course, $|T|$ is unity.

—NOTES—

A SECOND NOTE ON COMPRESSIBLE FLOW ABOUT BODIES OF REVOLUTION*

BY W. R. SEARS (*Cornell University*)

In a recent Note[1] the present author discussed the application of the linear-perturbation theory to the subsonic flow of a compressible fluid past a slender bod of revolution. It was shown that the several variants of the theory[2,3,4,5,6] lead to inconsistent results when applied, for example, to the flow about ellipsoids of revolution, and it was stated that this ambiguity results from applying the theory to a case that is essentially nonlinear. It was implied that all the variants are equally valid so far as the first-order theory is concerned.

In the present Note it will be shown that the various procedures are ***not***, in fact, equally acceptable, and that only one of them, in which the boundary condition is properly satisfied, can be considered to be correct, even to the first order.

The boundary-value problem involved is the following:

$$\left.\begin{aligned} &\beta^2\varphi_{xx} + (1/r)(r\varphi_r)_r = 0, \\ &\frac{\varphi_r}{U+\varphi_x} = \frac{dR}{dx} = R'(x), \quad \text{say, when} \quad r = R(x), \\ &\varphi \to 0 \quad \text{as either} \quad |x| \to \infty \quad \text{or} \quad r \to \infty. \end{aligned}\right\} \tag{1}$$

Here $\phi = Ux + \varphi$ is the velocity potential, x and r are the usual cylindrical coordinates, and $\beta^2 = 1 - M^2$, a constant defining the Mach number of the undisturbed flow. The body of revolution is defined by the function $R(x)$. The perturbation velocities φ_x and φ_r have been neglected in comparison with the free-stream velocity U; similarly, φ_x can be neglected in the first boundary condition in (1).

It is easily ascertained that solutions I, II, and III of the earlier Note[1] do not accurately satisfy this boundary condition. These solutions were derived from the incompressible-flow solution $f(x, r)$, which satisfies the following conditions:

$$\left.\begin{aligned} &f_{xx} + (1/r)(rf_r)_r = 0, \\ &f_r(x, S) = U\cdot dS/dx = US'(x), \quad \text{say;} \\ &f \to 0 \quad \text{as either} \quad |x| \to \infty \quad \text{or} \quad r \to \infty \end{aligned}\right\} \tag{2}$$

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* Received Sept. 18, 1946.

[1] W. R. Sears, ***On compressible flow about bodies of revolution***, Quart. Appl. Math. **4**, 191–193 (1946).

[2] S. Goldstein and A. D. Young, ***The linear perturbation theory of compressible flow with applications to wind-tunnel interference***, British Aero. Res. Com. Reports and Memoranda No. 1909 (1943).

[3] H. S. Tsien and L. Lees, ***The Glauert-Prandtl approximation for subsonic flow of a compressible fluid***, J. Aero. Sci. **12**, 173–187, 202 (1945).

[4] H. W. Liepmann and A. Puckett, ***Introduction to the aerodynamics of compressible fluids***, John Wiley and Sons, New York, 1946.

[5] R. Sauer, ***Theoretische Einführung in die Gasdynamik***, Springer, Berlin, 1943. Reprinted by Edwards Bros., Inc., Ann Arbor, 1945.

[6] B. Göthert, ***Ebene und räumliche Strömung bei hohen Unterschallgeschwindigkeiten***, Lilienthal Gesellschaft f. Luftfahrtforschung, Bericht **127**, 97–101 (1940).

In each solution, the body $R(x)$ is supposed to be related to the different body $S(x)$ in such a manner as to satisfy the boundary conditions. The various solutions are tabulated below, and in each case is stated the boundary condition that is actually satisfied:

I
$$\varphi(x, r) = \frac{1}{\beta} f(x, \beta r),$$
$$\varphi_r(x, S/\beta) = US'(x);$$

II
$$\varphi(x, r) = f(x, \beta r),$$
$$\varphi_r(x, S/\beta) = \beta US'(x);$$

III
$$\varphi(x, r) = f(x/\beta, r),$$
$$\varphi_r(x, S) = US'(x/\beta);$$

IV
$$\varphi(x, r) = \lambda f(x, \beta r),$$
$$\varphi_r(x, S/\beta) = \lambda\beta US'(x);$$

V
$$\phi = \beta^2 Ux + f(x, \beta r) = \beta^2 Ux + \varphi(x, r), \quad \text{say;}$$
$$\varphi_r(x, S/\beta) = U\beta S'(x) = (\beta^2 U)S'(x)/\beta.$$

It is seen that only Method V, and Method IV with $\lambda = 1/\beta^2$, satisfy in the x, r space the correct (linearized) boundary condition for flow about a solid body; i.e., that the stream surface has the same slope as the body *at the surface of the body*. Thus the ambiguity mentioned in the Note is eliminated and the formula for the maximum superstream velocity ratio on the surface becomes unique:

$$\frac{\phi_x - U}{U} = \frac{1}{\beta^2} F(\beta n), \tag{3}$$

where n is the ratio of maximum diameter to length for the particular family of bodies under consideration, and $F(n)$ is the maximum superstream velocity ratio in incompressible flow.

The reason for the confusion mentioned in the earlier Note, and undoubtedly contributed to by that Note, is not the nonlinear character of $F(n)$, but the fact that appreciable errors are introduced by use of the approximate boundary conditions shown in I–III above, *in the three-dimensional case*. In the analogous two-dimensional case, to be sure, the boundary values of the normal velocity component can be satisfied at an approximate location—it is customary to fix them along a straight line, for convenience—without introduction of any error in a first-order theory. Thus the variants, which differ only in the location where the slope is fixed, all produce the same result. This is not true, in general, in the three-dimensional case. This fact was overlooked by various writers[2,3,5] and by the present author in the earlier Note.[1]

The essential difference between the plane case and that of axial symmetry is understood by consideration of the distribution of singularities, along the axis, that is needed to produce the required flow. In the plane case, both the normal and the axial (downstream) velocity components u and v, are constant, to the first order, through a small distance each side of the surface of discontinuity; hence an approximation to the position at which the boundary values of v are fixed does not affect the

strength of singularities nor the resulting values of u. In the axially-symmetrical case, on the other hand, for any source-sink distribution along the axis, ϕ_r varies as $1/r$ while ϕ_x is nearly constant, for small r. Thus an approximation to the radius at which the boundary values of ϕ_r occur causes appreciable errors in ϕ_x.[7]

In conclusion, it now appears, after further study, that the conclusions of the earlier Note are in error, and that Method V, or its equivalent, is the only correct one for the case of axially-symmetric flow. Method V is that employed by Göthert,[6] and it must be agreed that his objections to the conventional procedure (Method II), which the author previously termed "fancied," are, in fact, valid.[8]

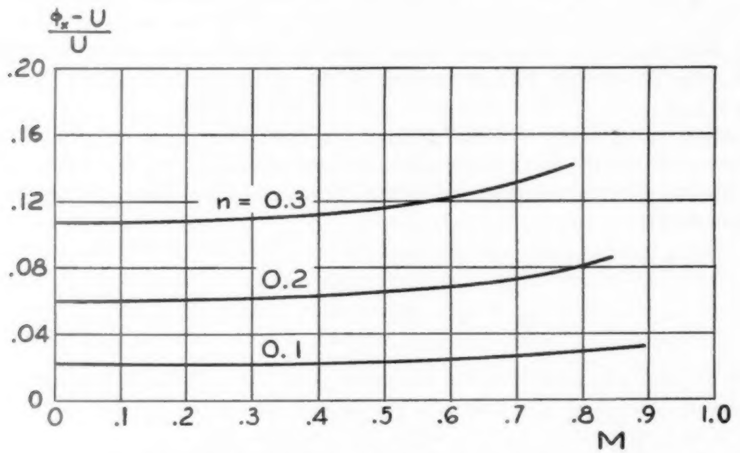

FIG. 1. The maximum superstream velocity ratio for ellipsoids of revolution in compressible flow. M is the free-stream Mach number and n is the ratio of maximum diameter to length of the ellipsoid.

For the particular case of ellipsoids of revolution, the results of Method V are shown in Figure 1. It should particularly be noted that the correct velocity-ratio formula, equation (3), does not permit a universal velocity-correction curve to be drawn as function of the Mach number, as the result of Method I would, for example, and as in the two-dimensional case. The effect of increasing M is seen to depend intimately upon n and upon the nature of $F(n)$; hence the curves of Figure 1 cannot be expected to apply to bodies other than the ellipsoids of revolution.

[7] In the paper of Goldstein and Young (*loc. cit.*) this approximation is introduced in equation (21) for example, when y is replaced by h in integrating to determine $y-h$.

[8] *Note added in proof:* In a report recently released [L. Lees, *A discussion of the application of the Prandtl-Glauert method to subsonic compressible flow over a slender body of revolution*, N.A.C.A. Techn. Note No. 1127], one of the authors mentioned here has reconsidered the problem and has arrived at conclusions in complete agreement with those of the present note.

A NOTE ON STABILITY CALCULATIONS AND TIME LAG*

By SEYMOUR SHERMAN (*University of Chicago*)

In recent technical literature[1] covering widely different fields, investigations have appeared of the zeros of particular exponential sums, one example of which is

$$az^2 + bz + \beta z e^{-z} + c. \tag{1}$$

The question is: what are the conditions on a, b, β, and c which are necessary and sufficient that the real parts of all the roots be negative, thus indicating stability. There also have appeared papers in pure mathematics[2] which discuss similar problems and which supply useful techniques for their solutions. It is the purpose of this note to indicate how one such technique, which shall be referred to as the Cauchy-Sturm method, may be applied to a discussion of the zeros of transcendental expressions such as (1).

Equation (1) arises in the study of control systems with retarded action or time lags.[3] Several attempts have been made to study the zeros of this function and the results have not been consistent. Minorsky,[4] in one of his papers, expands the function in a power series

$$\begin{aligned} f(z) &= az^2 + bz + \beta z e^{-z} + c \\ &= (c + (b + \beta)z + (a - \beta)z^2 + \frac{\beta z^3}{2!} - \frac{\beta z^4}{3!} + \frac{\beta z^5}{4!} - \frac{\beta z^6}{5!} + \cdots . \end{aligned}$$

He then attempts to approximate the zeros of $f(z)$ by taking zeros of partial sums. For nonzero c and β we can choose a partial sum of degree $n(n \geqq 3)$ such that c and $(-1)^{n-1}\beta$ have opposite signs and so the partial sum

$$c + (b + \beta)z + (a - \beta)z^2 + \frac{\beta z^3}{2!} + \cdots + (-1)^{n-1} \frac{\beta z^n}{(n-1)!}$$

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* Received March 28, 1946.

[1] H. Bateman, *The control of an elastic fluid*, Bull. Amer. Math. Soc. **51**, 601–646 (1945), especially pp. 618–626. Further references given there.

A. Callender, D. R. Hartree, A. Porter, *Time-lag in a control system*, Trans. Roy. Soc. London (A) **235**, 415–444 (1936).

A. Callender, and A. B. Stevenson, Proc. Soc. Chem. Industry (Chem. Engg. Group) **18**, 108 (1936).

D. R. Hartree, A. Porter, A. Callender, A. B. Stevenson, *Time-lag in a control system*, Proc. Roy. Soc. London (A) **161**, 460–476 (1937).

N. Minorsky, (i) *Control problems*, J. Franklin Inst. **232**, 519–551 (1941), especially pp. 524–529.

N. Minorsky, (ii) *Self-excited oscillations in dynamical systems possessing retarded actions*, J. Appl. Mech. **64**, A65–A71 (1942).

F. Reinhardt, *Parallelbetrieb von Synchrongeneratoren mit Kraftsmaschinenregelung konstanter Verzögerungszeit*, Wissenschaftl. Veröffentl. Siemens Werke **18**, 24–44 (1939).

[2] R. E. Langer, *On the zeros of exponential sums and their integrals*, Bull. Amer. Math. Soc. **37**, 213–239 (1931). Further bibliography can be found there.

[3] For *example* the roots of (1) determine the stability of the following system with retarded viscous term:

$$az''(t) + bz'(t) + \beta z'(t-1) + c = 0,$$

where $z(0)$ is given and $z'(t) = 0$, $-1 < t \leqq 0$.

[4] N. Minorsky, (i) supra.

would have at least one positive zero. This would seem to suggest that dynamical systems with retarded viscous terms are necessarily unstable. However, from a theorem of function theory,[5] we know

For every power series, every point of the circle of convergence is a limit point of zeros of partial sums.

Although function (1) is entire and so has no circle of convergence, the theorem stated above shakes our faith in the value of approximating the zeros of function by taking the zeros of partial sums of its Taylor series.

Reinhardt,[6] in his discussion of the same equation, first considers a particular case of equation (1), namely

$$z^2 + .5z + .5ze^{-\pi z} + 1 = 0, \tag{2}$$

and among the infinite number of zeros of $f(z)$ chooses one which has the largest real part and small imaginary part (most unstable and corresponding to low frequency). Arguing that this root, corresponding as it does to a low resonant frequency, is physically the most significant, Reinhardt studied the zeros of (1) which for other choices of parameters could be expanded in a series about the "original" zero. Thus he studied one of the infinite number of zeros of (1) intensively and later made approximations to the others. He discovered that for some choices of a, b, β, and c this root had a positive real part and for other choices of these parameters this root had a negative real part. Thus, the results of Reinhardt and Minorsky are inconsistent. Since both of their arguments are approximate a further study is indicated. Minorsky has published another analysis[7] of this subject which allows for the possibility of stability.

A method frequently useful for counting the zeros of an analytic function $f(z)$ in a simply connected domain D bounded by curve C is Cauchy's[8] index theorem:

If $w=f(z)$ is an analytic function of z in a simply connected domain D bounded by a closed curve C, $f(z)\neq 0$, $z\in C$, and z traverses C in a counterclockwise direction, then $f(z)$ will traverse a closed curve in the w-plane and the number of zeros of $f(z)$ in D is equal to the number of times the w-contour encircles the origin.

This theorem is at the heart of Nyquist's[9] criterion for the stability of amplifiers and Routh's[10] stability criterion. An attempt will be made to apply Cauchy's Theorem to the zeros of (1). We first note that as z traverses C in a counterclockwise sense w may cross the real axis. Let γ be the number of times w crossed the real axis in a counterclockwise direction relative to the origin (i.e., from quadrant IV to quadrant I or from quadrant II to quadrant III) and let α be the number of times w crossed the real axis in a clockwise direction relative to the origin (i.e., from quadrant I to quadrant IV or from quadrant III to quadrant II). The number of zeros of $f(z)$ in D is then equal to $1/2(\gamma-\alpha)$.

Conditions are sought on a, b, β, and c (all nonzero) in order that all of the roots of

$$w(z) = az^2 + bz + \beta ze^{-z} + c = 0$$

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[5] E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, 1939, p. 238.

[6] See (1) supra.

[7] N. Minorsky, (ii) supra.

[8] H. W. Bode, *Network analysis and feedback amplifier design*, D. Van Nostrand Co., 1945, chap. 8. See Titchmarsh, supra pp. 115–116.

[9] See Bode supra.

[10] See E. J. Routh, *Dynamics of a system of rigid bodies*, Macmillan and Co., 1892, Part II, pp. 191–202.

have negative real parts. We assume that $w(z)$ has no zeros on the imaginary axis.

Our domain D is the semicircle

$$D:\ \mathfrak{R}(z) > 0, \qquad |z| < R$$

in the z-plane.

For $R(>0)$ sufficiently large, if $|z| \geqq R$ and[11] $\mathfrak{R}(z) \geqq 0$, then

$$|az^2| > |\beta z e^{-z} + bz + c|.$$

Since $az^2 \neq 0$ for $|z| \geqq R$, $\mathfrak{R}(z) \geqq 0$, we have, by arguing from Rouche's theorem,[12] that $w(z) \neq 0$, for z in this region. If we choose the R given above, all the zeros of $w(z)$ lying in the right half-plane will lie in D and so we apply Cauchy's Index Theorem to this region. The boundary curve C may be broken into two parts

$$A;\ \mathfrak{R}(z) = 0 \qquad |z| \leqq R$$

and

$$B;\ \mathfrak{R}(z) > 0 \qquad |z| = R.$$

We consider A for large R. Let $z = iy$ and

$$w(iy) = -ay^2 + c + \beta y \sin y + i(by + \beta y \cos y).$$

Note that $\mathfrak{R}(w)$ is an even function of y and $\mathfrak{I}(w)$ is an odd function of y.

In practical application frequently $a>0$, $c>0$. Let us consider the special case $b \geqq |\beta| > 0$. If $y=0$, then $w(0)=c>0$. If $0<y\leqq R$, then $\mathfrak{I}(w)=y(b+\beta \cos y) \geqq 0$ and w is in either quadrant I or quadrant II. For large R, $w(iR)$ is in the second quadrant. If $-R \leqq y < 0$, then $\mathfrak{I}(w)=y(b+\beta \cos y) \leqq 0$ and w is in either the third or fourth quadrants. Thus, as z traverses A from $+iR$ to $-iR$, w crosses the real axis once in a clockwise direction relative to the origin (from quadrant I to IV). On the other hand, for large R, $w(z)$ is dominated by az^2 and as z traverses B from $-iR$ to $+iR$, the net number of times that w crosses the real axis is just once in a counterclockwise direction relative to the origin (from quadrant IV to I). Since $1/2(\gamma-\alpha)=0$ for C, $f(z)$ has no zeros in D, therefore all the zeros of $f(z)$ have negative real parts. It should be noted that if we remove the restriction of $c>0$, then[13] $\gamma-\alpha=1-\operatorname{sgn}[c(b+\beta)]$.

We may rephrase the results as follows: *In a one degree of freedom mechanical system with positive mass a, positive spring constant c and positive damping coefficient b, and with retarded (unit time lag) coefficient β, if the damping coefficient is greater than or equal to the absolute value of the retarded damping coefficient, then the system is stable.*[14]

Suppose we relax the restrictions of the preceding two paragraphs but still require $a>0$ and consider (1) on curve C. In order to compute $1/2(\gamma-\alpha)$ for the line

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[11] If $z=x+iy$, x, y real, then

$$\mathfrak{R}(z) = x \quad \text{and} \quad \mathfrak{I}(z) = y.$$

[12] See Titchmarsh, p. 116.

[13] sgn (x) is a real valued function of a real variable defined as follows:

$$\operatorname{sgn}(x) = \begin{cases} -1, & x \quad 0, \\ 0, & x = 0, \\ 1, & x \quad 0. \end{cases}$$

[14] This is consistent with Minorsky (ii) p. A69. Note that if $b \leqq -|\beta| < 0$ (the damping coefficient less than or equal to the negative of the absolute value of retarded damping coefficient), then the system is unstable with two zeros of (1) in the right half-plane.

segment A and arc B we wish first to note where w crosses the real axis. Because $\mathfrak{R}(w(iy))$ and $\mathfrak{I}(w(iy))$ are even and odd functions of y respectively, we need only consider those positive values of y for which $\mathfrak{I}(w(iy)) = y(b+\beta\cos y) = 0$.

If we temporarily ignore the case previously considered where $y(b+\beta\cos y)$ had no positive zeros, but assume merely that $\beta \neq 0$, then the positive roots of $y(b+\beta\cos y)$ will be of the form

$$\operatorname{arc\,cos}\left(\frac{-b}{\beta}\right) + 2m\pi, \qquad m = 0, 1, 2, \cdots,$$

where we permit arc cos to take two positive values between 0 and 2π (including the latter). We number the positive zeros of $y(b+\beta\cos y)$ in order of increasing size $y_1, y_2, \cdots$. Let y_M be such a positive zero of $b+\beta\cos y$ that

(1) $\beta \sin y_M < 0$

and

(2) $-ay^2+c+\beta y \sin y < 0$ for[15] $y \geqq y_M$.

We might, if we so wished, have chosen y_M so large that $y_M < \mathrm{R} < y_{M+1}$. For the purposes of our subsequent calculation this would be inconvenient, but since it is convenient for the purpose of the argument, we shall make this assumption during the proof. We now calculate $\gamma-\alpha$ for curve C. In other words we have to consider the number of times and the direction relative to the origin in which w crosses the real axis as z traverses C in a counterclockwise direction relative to D. The only values of z along A for which w hits the real axis are: 0, $\pm y_j$; $j = 0, 1, 2, \cdots, M$. Let us consider y_j, which is by definition positive. The contribution to $\gamma-\alpha$ due to this crossing is

$$\begin{aligned}
&- \operatorname{sgn}\,(-ay_j^2 + c + \beta y_j \sin y_j)\operatorname{sgn}\left[\frac{d}{dy}\{y(b+\beta\cos y)\}\right]_{y=y_j} \\
&= -\operatorname{sgn}\,[-ay_j^2 + c + \beta y_j \sin y_j]\operatorname{sgn}\left[\frac{d}{dy}(b+\beta\cos y)\right]_{y=y_j} \\
&= +\operatorname{sgn}\,[(-ay_j^2 + c + \beta y_j \sin y_j)(\beta \sin y_j)].
\end{aligned}$$

Because $\mathfrak{R}(w(iy))$ and $\mathfrak{I}(w(iy))$ are, respectively, even and odd functions of y, the contribution to $\gamma-\alpha$, because of the crossing at $z = -iy_j$ is also

$$\operatorname{sgn}\,[(-ay_j^2 + c + \beta y_j \sin y_j)(\beta \sin y_j)].$$

The definition of y_M was so arranged that the net contribution to $\gamma-\alpha$ because of the crossing of the real axis corresponding to $z=0$ and those crossings corresponding to z on B totals to $1-\operatorname{sgn}[c(b+\beta)]$, as in the case $a>0$, $b \geqq |\beta| > 0$, previously considered. We now have[16]

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[15] If $y > (2c/2\sqrt{ac}-\beta) > 0$, then $-ay^2+c+\beta y \sin y < 0$. Thus in many cases the condition $y > (2c/2\sqrt{ac}-\beta) > 0$ may be used rather than (2). There are other approximations to (2) which might prove convenient in different cases.

[16] If $\mathfrak{I}[w(iy)] \neq 0$ for any real value of y or if $\mathfrak{I}[w(iy)] = 0$ implies that $\sin y = 0$, then substitute 0 for $2\sum_{j=1}^{M} \operatorname{sgn}\{[-ay_j^2+c+\mathrm{b}y_j \sin y_j][\beta \sin y_j]\}$.

$$(\gamma - \alpha) = 1 - \operatorname{sgn}\,[c(b + \beta)] + 2\sum_{j=1}^{M} \operatorname{sgn}\,\{[-\,ay_j^2 + c + \beta y_j \sin y_j][\beta \sin y_j]\}.$$

Thus a one degree of freedom mechanical system with positive mass a, and nonzero spring constant c, damping constant b, retarded (unit time lag) damping constant β, and with $w(iy) \neq 0$, y real, is stable if, and only if,[16]

$$1 - \operatorname{sgn}\,[c(b + \beta)] + 2\sum_{i=1}^{M} \operatorname{sgn}\,\{[-\,ay_i^2 + c + \beta y_i \sin y_i][\beta \sin y_i]\} = 0. \qquad (*)$$

Only minor modifications are required in order to take care of the case $a<0$ or the degenerate cases where one or more of the coefficients is zero. The expression (*) may readily be calculated since it involves only four readings from a trigonometric table and subsequent evaluations of the sign of quadratic expressions.

For dynamical systems with a larger number of degrees of freedom or with more lag terms, we get higher powers of z or more exponential terms and the application is complicated but not hopeless. A machine[17] of the isograph type should prove helpful where extended calculations on complicated systems are being considered.

Stability calculations are always easier than control calculations. The usual design procedure would be to discover a range of a, b, β, and c corresponding to stable responses and then to investigate the detailed response for a few choices of a, b, β, and c. Thus we would expect that for any particular choice of a, b, β, and c the control calculation will be more laborious. In the stability calculation of this paper we have been, in effect, investigating the relations between the parameters a, b, β, and c and the asymptotic character of the solutions of

$$az''(t) + bz'(t) + \beta z'(t-1) + cz(t) = 0,$$

subject to the boundary conditions:

$$z(0) \neq 0, \quad \text{and} \quad z'(t) = 0, \quad -1 < t \leqq 0.$$

For many design purposes information more specific than the asymptotic character of the oscillation might be needed. For instance, after having chosen a, b, β, and c so that the transient oscillations are stable (all of the roots of (1) have negative real parts), one may be interested in the detailed response $z(t)$ of the system to a given impressed force $f(t)$. This is the control problem. We therefore seek the solution of

$$az''(t) + bz'(t) + \beta z'(t-1) + c = f(t),$$

subject to the boundary conditions:

$$z(0) \ \text{given}, \quad z'(t) = 0, \quad -1 < t \leqq 0.$$

We consider first the response of the system in the first second. During that time it will act like a classical one degree of freedom system with constant mass, viscosity, and elastic coefficients a, b, and c, and variable impressed force $f(t)$. The response is given by the solution of

$$az''(t) + bz'(t) + cz(t) = f(t),$$

[17] See T. C. Fry, *Some numerical methods for locating roots of polynomials*, Quart. Appl. Math. **3**, 89–105 (1945), especially pp. 100–103.

where $z_1(0)$, given and $z_1'(0)=0$. The solution of this equation can be found in any standard text[18] on differential equations, but, depending on the nature of $f(t)$, might best be done by numerical integration. In any event, we have $z(t)=z_1(t), 0\leqq t\leqq 1$. Now during the second second we can again consider our system as a one degree of freedom system with the same mass, viscosity, and elastic coefficients, but with an impressed force which depends on $f(t)$ and velocity at time $t-1$. Consider the equation

$$az_2''(t) + bz_2'(t) + cz(t) = f(t+1) - \beta z_1'(t),$$

for $0\leqq t\leqq 1$, where $z_2(0)=z_1(1)$, $z_2'(0)=z_1'(1)$, and $z_1'(t)$ are derived from the solution of the previous equation. Again by standard methods we solve for $z_2(t)$, $0\leqq t\leqq 1$. This is related to the actual response $z(t)$ during the second second by

$$z(t) = z_2(t-1), \qquad 1 \leqq t \leqq 2.$$

We can continue this process[19] for larger values of t if so desired.

THE CENTER OF SHEAR AND THE CENTER OF TWIST*

By A. WEINSTEIN (*Carnegie Institute of Technology*)

In two recent papers W. R. Osgood[1] and J. N. Goodier[2] reconsider the much discussed question of the center of shear and center of twist, the former author pointing out the disagreement in the literature as to the location of the center of shear. However no mention is made of the important paper by P. Cicala[3] which, together with a paper of Trefftz,[4] will form the basis of the following remarks.

R. V. Southwell[5] has clearly pointed out that the two centers, which are intuitively well known to engineers, constitute two different concepts and are not just synonyms for the same point. The center of twist is the point at rest in every section of a uniform beam subject to a twist by a terminal couple and rigidly clamped at the other end. The center of shear (called also flexural center) is the point at which an applied shearing force would produce a flexure without torsion. However, neither of these points can be explicitly computed, since the displacements of a rigidly clamped beam under torsion are not known and, on the other hand, the concept of flexure without torsion is still to be exactly defined. Nevertheless Southwell, using Maxwell's reciprocal relations in a summary way, makes plausible the coincidence of both centers.

As Goodier points out, Saint Venant's theory of torsion and flexure of beams does

[18] E. L. Ince, *Ordinary differential equations*, Dover Publications, 1944, chap. 6.

[19] Analogous stepwise integration can be found in R. M. Head, *Lag determination of altimeter systems*, J. Aeronaut. Sci. **12**, 85–93 (1945) and C. C. Kennedy, *Measuring the Coulomb and viscous components of friction*, Instruments **15**, 404–410 (1942).

* Received May 9, 1946.

[1] W. R. Osgood, J. Appl. Mech. **10**, A-62–A-64 (1943).

[2] J. N. Goodier, J. Aeron. Sci. **11**, 272–280 (1944).

[3] P. Cicala, Atti R. Acc. Sci. Torino **70**, 356–371 (1935).

[4] E. Trefftz, Z. angew. Math. Mech. **15**, 220–225 (1935).

[5] R. V. Southwell, *An introduction to the theory of elasticity*, 2nd ed., 1941, p. 29.

not give any definition for either point. Several suggestions have been made concerning the use of Saint Venant's theory for this purpose, and it will be shown here that a definition can be given, which is based on Saint Venant's formulae and leads to an explicit proof of the coincidence of both centers.

We shall accept Trefftz' definition of the center of shear, and refer the reader to Trefftz' paper which also contains a clear exposition of the theory of flexure. It will suffice to mention here that Trefftz has been led to his definition by the reciprocity law of Maxwell and Betti.

Let z be the axis of a uniform beam and x, y the principal directions in the cross section S, the origin O being the centroid of the cross section. According to Trefftz, the coordinates x_F and y_F of the center of shear are given by

$$x_F = -\frac{1}{I_x}\iint_S y\varphi(x, y)dxdy, \qquad y_F = \frac{1}{I_y}\iint_S x\varphi(x, y)dxdy, \tag{1}$$

where $\varphi(x,y)$ denotes the warping function for the cross section in the theory of torsion and

$$I_x = \iint_S y^2dxdy, \qquad I_y = \iint_S x^2dxdy$$

are the moments of inertia with respect to the axes of x and y.

Turning now to the center of twist, the definition which will be given here will be based on an idea of Cicala, who however fails to recognize the coincidence of the two points as postulated by Southwell.

According to Saint-Venant, the displacements of the twisted beam are given by

$$\begin{aligned} u &= \tau(-zy + a + qz - ry), \\ v &= \tau(zx + b + rx - pz), \\ w &= \tau[\varphi(x, y) + c + py - qx], \end{aligned} \tag{2}$$

where τ denotes the infinitesimal angle of twist. The last three terms in each formula represent a rigid body displacement. We have only six constants a, b, c, p, q, r at our disposal, so that a rigid clamping cannot be obtained. However, an approximate clamping of the end section $z=0$ can be obtained by adopting the following postulates.

(*i*) The displacements u and v in the section $z=0$ are zero:

$$u = 0, \qquad v = 0 \quad \text{for} \quad z = 0. \tag{3}$$

(*ii*) The mean square of the displacement w in the z direction is a minimum with respect to the parameters c, p and q which occur in the last Eq. (2):

$$\iint_S w^2dxdy = \text{minimum}. \tag{4}$$

Denoting this integral by J, we can write in place of (4)

$$J(c, p, q) = \text{minimum with respect to } c, p \text{ and } q. \tag{4'}$$

From (2) and (3) we obtain

$$a = 0, \qquad b = 0, \qquad r = 0,$$

so that we have, for any section $z = \text{const.}$,

$$u = \tau z(q - y), \qquad v = \tau z(x - p). \tag{5}$$

We see that, in every section, u and v vanish at the point $x = x_T$, $y = y_T$, where

$$x_T = p, \qquad y_T = q. \tag{6}$$

This point will be, by definition, the center of twist. In order to compute p and q, we use (4) or (4'), and set the derivatives of $J(c, p, q)$ with respect to c, p and q equal to zero. In this way we obtain the following equations.

$$\iint_S w\,dxdy = 0, \qquad \iint_S wy\,dxdy = 0, \qquad \iint_S \psi x\,dxdy = 0, \tag{7}$$

Using the last Eqs. (2), namely $w = \tau(\varphi(x, y) + c + py - qx)$, and observing that

$$\iint_S y\,dxdy = \iint_S x\,dxdy = 0,$$

the origin being the centroid of the cross section, we obtain from (7) the three equations

$$\iint_S (\varphi + c)\,dxdy = 0, \qquad \iint_S (\varphi + py)y\,dxdy = 0, \qquad \iint_S (\varphi - qx)x\,dxdy = 0. \tag{8}$$

The first of these equations yields the value of the constant c, while the second and the third gives us the coordinates $p = x_T$ and $q = y_T$ of the center of twist, namely

$$x_T = -\frac{1}{I_x}\iint_S y\varphi(x, y)\,dxdy, \qquad y_T = \frac{1}{I_y}\iint_S x\varphi(x, y)\,dxdy.$$

A comparison with (1) shows that the center of shear coincides with the center of twist.

Incidentally, the first Eq. (7), namely $\iint_S w\,dxdy = 0$, shows that the average displacement in the direction of the axis of the beam is zero.

It is interesting to note that both centers, as defined here, are independent of Poisson's ratio.

ELASTIC STRESSES DUE TO A SEMI-INFINITE BAND OF HYDROSTATIC PRESSURE ACTING OVER A CYLINDRICAL HOLE IN AN INFINITE SOLID*

By O. L. BOWIE (*Watertown Arsenal, Watertown, Mass.*)

Recently C. J. Tranter[1] has presented expressions for the stresses due to a finite band of hydrostatic pressure acting over part of the length of a cylindrical hole which extends through an infinite elastic solid. By a similar analysis, expressions for the stresses due to a semi-infinite band of hydrostatic pressure acting over the cylindrical hole can be found. Referring to the analysis by Tranter, it is evident that the only change in boundary conditions is that involving σ_r at $r=a$.

A semi-infinite band of hydrostatic pressure, p, is equivalent to the sum of

$$\sigma_r = -\frac{p}{2}, \qquad \begin{cases} -\infty < z < \infty \\ r = a \end{cases} \tag{1}$$

and

$$\left.\begin{aligned} \sigma_r &= -\frac{p}{2}, & -\infty < z < 0 \\ &= \frac{p}{2}, & 0 < z < \infty \end{aligned}\right\} r = a. \tag{2}$$

The stresses due to the loading (1) are well known,[2] whereas, the analysis corresponding to (2) may be carried out in a manner very similar to that used by Tranter. The combined stress distributions determined for (1) and (2) yield the following expressions for the stresses:

$$\sigma_r = -\frac{p}{2}\left(\frac{a^2}{r^2}\right) - \frac{pa}{\pi r}\int_0^\infty \Big[\alpha\rho K_0(\alpha)K_0(\rho) + \alpha K_0(\alpha)K_1(\rho) - \rho K_0(\rho)K_1(\alpha) - \left\{\rho^2 + 2(1-\nu)\right\}K_1(\alpha)K_1(\rho)\Big]\cdot\frac{1}{\alpha D(\alpha)}\sin(z\alpha/a)\,d\alpha$$

$$\tau_{rz} = +\frac{p}{\pi}\int_0^\infty \left[\alpha K_0(\alpha)K_1(\rho) - \rho K_0(\rho)K_1(\alpha)\right]\cdot\frac{1}{D(\alpha)}\cos(z\alpha/a)\,d\alpha$$

$$\sigma_\theta = \frac{p}{2}\left(\frac{a^2}{r^2}\right) + \frac{pa}{\pi r}\int_0^\infty \Big[\alpha K_0(\alpha)K_1(\rho) + (2\nu-1)\rho K_0(\rho)K_1(\alpha) - 2(1-\nu)K_1(\alpha)K_1(\rho)\Big]\cdot\frac{1}{\alpha D(\alpha)}\sin(z\alpha/a)\,d\alpha$$

$$\sigma_z = \frac{p}{\pi}\int_0^\infty \left[\alpha K_0(\alpha)K_0(\rho) + 2K_0(\rho)K_1(\alpha) - \rho K_1(\alpha)K_1(\rho)\right]\cdot\frac{1}{D(\alpha)}\sin(z\alpha/a)\,d\alpha$$

where the notation is that used by Tranter.

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* Received Sept. 27, 1946.

[1] C. J. Tranter, *On the elastic distortion of a cylindrical hole by a localized hydrostatic pressure*, Quart. Appl. Math. 4, 298 (1946).

[2] A. H. Love, *Theory of elasticity*, Cambridge Press, 1927, p. 144.

It is evident that by a proper superposition of the preceding results, Tranter's expressions for a finite band of pressure may be obtained immediately. A similar variation on Rankin's solution[3] for the case of a finite band of external pressure acting on a solid cylinder will yield expressions for the stresses corresponding to a semi-infinite band of external radial pressure acting on a solid cylinder.

ON A CONFORMAL MAPPING TECHNIQUE*

By G. F. CARRIER (*Brown University*)

1. Introduction. Many physical problems are readily reduced to the problem of finding the value of a harmonic function on a smooth closed curve corresponding to given boundary conditions and certain interior singularities. For example, it has been shown [1] that the problem of finding the flow of an incompressible non-viscous fluid past a periodic array of airfoils[1] may be replaced by that of determining the flow within a smooth closed stream-line C with a source-vortex and a sink-vortex at specified interior points.[2] Analogous problems can obviously occur in other physical situations which lead to the Laplace equation.

Here we shall develop a method of mapping a smooth closed curve C conformally onto the unit circle so as to carry two arbitrarily specified interior points into the points $\pm a$. The mapping is continuous within and on C.

Since the solution requires the integration of a single non-homogeneous Fredholm integral equation, we also present a numerical procedure which is useful in solving such equations. The over-all procedure seems preferable, in general, to Theodorsen's method of mapping "nearly circular regions" [2], [3], since the equation used here has a non-singular kernel in contrast to those in his two simultaneous integral equations.

2. The integral equation. Let us consider the problem of finding that complex potential $F(\zeta)=\phi+i\psi$ on the closed curve C[3] (see Fig. 1) which corresponds to two equal and opposite interior logarithmic singularities at the points P and Q and let ψ vanish on C. We may write then, for points on and within C,

$$F(\zeta) = \ln [(\zeta - P)/(\zeta - Q)] + f(\zeta), \tag{1}$$

where f is analytic within C and continuous on C. In principle, no difficulty, would be encountered if the unit coefficients of these singularities were replaced by complex numbers, but the present form is sufficiently general for our purpose.

Consider the integral

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[3] A. W. Rankin, *Shrink-fit stresses and deformations*, J. Appl. Mech. **11**, A-77 (1944).

* Received Sept. 13, 1946.

[1] The airfoil shape being arbitrary.

[2] C, in this case, is a non-self-intersecting convex curve with a continuously turning tangent. By "smooth" we shall henceforth imply such a curve.

[3] C is any closed, not self-intersecting, curve.

$$I(\zeta_0) = \oint_S \frac{F(\zeta)}{\zeta - \zeta_0} d\zeta \equiv 0, \tag{2}$$

where S is that closed contour (Fig. 1) which contains no singularities. As a conse-

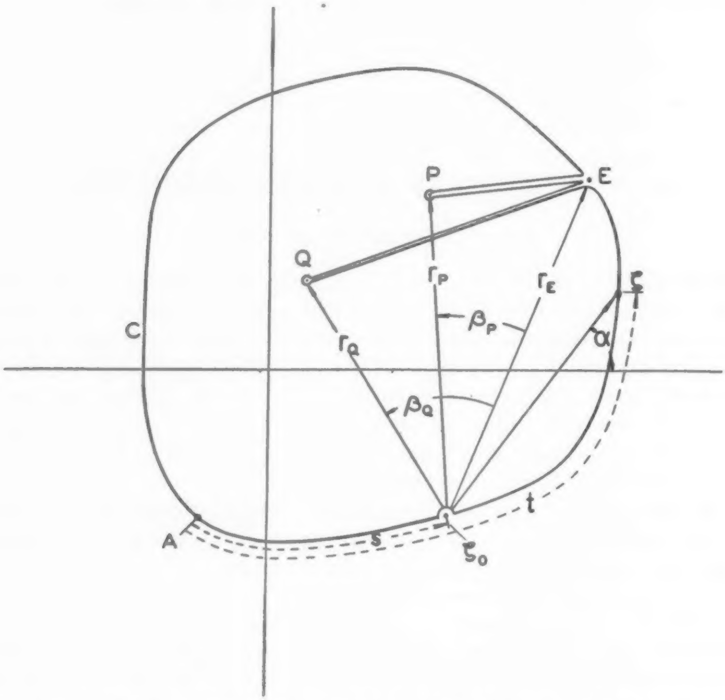

FIG. 1. A is an arbitrarily chosen origin of the coordinates s, t; the point E may also be taken arbitrarily.

quence of the fact that the logarithmic terms differ by $2\pi i$ on opposing sides of the cuts, the integral of Eq. (2) may be written in the form

$$- \pi i F(\zeta_0) + \int_{\zeta_0^+}^{\zeta_0^-} F(\zeta) d\{\ln (\zeta - \zeta_0)\} + 2\pi i \ln \frac{P - \zeta_0}{Q - \zeta_0} = 0, \tag{3}$$

provided the limiting process is performed wherein the radii of the circles about P, Q, and ζ_0, and the width of the cuts, are made to tend to zero. Since $\psi = 0$ on C, the imaginary parts of the foregoing equation lead to the integral equation

$$\phi(s) = \frac{1}{\pi} \int_C \phi(t) d\alpha + 2 \ln r_p/r_q. \tag{4}$$

The symbols not used previously are defined in Fig. 1. The angle α is obviously a function of s and t, and its increment in Eq. (4) corresponds to an increment dt. This equation has been obtained previously by a somewhat less straightforward method [1]

wherein the singularities were taken with complex coefficients.[4] In this case, Eq. (4) has terms $-\Gamma_p\beta_p/\pi$ and $-\Gamma_q\beta_q/\pi$.

A method of solving this integral equation will be presented in Sec. (4). Note that for a circle, the kernel is constant ($\partial\alpha/\partial\theta=\frac{1}{2}$, where $\zeta=e^{i\theta}$), and the solution is already obtained. A closely analogous situation was previously obtained by Prager [4].

3. The mapping problem. We now turn to the problem of mapping C on the unit circle in the z plane so that P, Q go into the points $\pm a$. The mapping is to be regular within and on C. The magnitude of the real quantity a cannot be arbitrarily chosen but will follow from the analysis. We first show that such a mapping exists. Two facts are well known: (1) there exists a mapping such that the closed curve C goes conformally into the unit circle in the z_1 plane and such that P, Q go into the (unspecified) points P', Q'; (2) there exists a bilinear transformation such that the unit circle goes into the unit circle and any two interior points P', Q', go into $\pm a$ (a being determined by the values of P', Q'). Thus the mapping exists.

Now consider that the integral equation of the foregoing section has been solved for $\phi(s)$. Denote its minimum value by $\phi_{\min}$.

Note that for the points $z=e^{i\theta}$,

$$F_2(z) = \phi_2(\theta) + iO = \ln \frac{-(z-a)\left(z-\dfrac{1}{a}\right)}{(z+a)\left(z+\dfrac{1}{a}\right)} + K(a) \tag{5}$$

is the solution of the potential problem wherein $\psi=0$ on the unit circle and unit singularities are placed at $\pm a$. The real number $K(a)$ may be chosen so that the smallest value taken by $\phi_2(\theta)$ is $\phi_{\min}$.[5]

It is evident that when the curve C is mapped into the z plane in the specified manner, the potentials ϕ, ϕ_2, in the two planes must have the same value at corresponding points, i.e. at points $z(\zeta_j)$ and ζ_j. Thus, a must be chosen such that the greatest value of ϕ is equal to the greatest value of ϕ_2. Once this has been done, the mapping function $z(\zeta)$ along C is implied by the relation

$$\phi_2(\theta) = \phi(s). \tag{6}$$

One may now quickly find $z(\zeta)$ along C, and, if it is required, find $z(\zeta)$ at any interior point by using the Cauchy formula

$$2\pi i z(\zeta_i) = \oint_C \frac{z(\zeta)}{\zeta-\zeta_i}\, d\zeta. \tag{7}$$

The physical problem may now be treated by the conventional methods used when conformal mapping is applied. In particular, for the fluid mechanics problem mentioned previously, complex singularities may be placed at $\pm a$, the imaginary coefficient of the singularity corresponding to the point infinitely far downstream in

[4] In [1], the vertex terms have the incorrect sign.

[5] Actually $K(a)$, a, are determined simultaneously by this condition and that appearing in the next paragraph.

the physical plane being determined by the Joukowski condition.[6] The velocity distribution along the vane profile is determined using only the analytic[7] transformations which led to C and the function $d\theta/ds$.

If it is only necessary to place one singular point P within the unit circle, it may be mapped into the origin. One uses doublets at P and at the origin. The strength of the doublet must be adjusted as was a in the preceding problem.

4. The solution of the integral equation. An elementary numerical procedure which resembles the relaxation process conventionally applied to differential equations [5] is useful in solving Eq. (4). We choose several points s_k (or t_k) roughly uniformly spaced along C, compute $\alpha(s_k, t_n) = \alpha_{kn}$ for each k and n, and determine the quantities $\Delta_{kn} = (\alpha_{k,n+1} - \alpha_{k,n-1})/2\pi$. We now write Eq. (4) in the form

$$\phi(s_k) \simeq \sum_n \phi(t_n)\Delta_{kn} + 2 \ln \frac{r_p(s_k)}{r_q(s_k)} \cdot \tag{8}$$

and guess the values of $\phi(t_n)$. Using any convenient arrangement, we compute $\phi(s_k)$ and compare the computed and guessed results. We then make a better guess for the quantities $\phi(t_n)$ [according to the form of the Δ_{kn} and to any experience gained in previous attempts, computing the change in $\phi(s_k)$] in such a manner that the two sets of ϕ values become essentially alike. The accuracy which can be obtained is a function of the fineness of the spacing of points in the kernel.

The usual iteration procedures which are applied to Fredholm equations, or to the systems of algebraic equations to which they may be reduced, differ from the foregoing in that no freedom of choice is allowed beyond the first approximation. In this method, the higher approximations may lead to much more rapid convergence than such iterations just as in the relaxation method in the numerical solution of differential equations.

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[6] $\partial\phi/\partial\theta$ must vanish at the point on $|z| = 1$ into which the trailing edge of the vane was mapped.

[7] By "analytic" we imply functions in closed form as contrasted with those obtained by numerical procedures.

ON HYPERSONIC SIMILITUDE*

By WALLACE D. HAYES (*North American Aviation, Inc.*)

A recent paper by H. S. Tsien[1] presents a law of similitude for two-dimensional potential hypersonic flows over a slender body. Hypersonic flow is fluid flow for which the Mach number is much greater than one. If the transformation

$$x = b\xi, \quad (1a) \qquad y = \delta\eta, \quad (1b) \qquad \phi = a_0\delta f(\xi, \eta) \quad (1c)$$

is made, where b and δ are the length and thickness of the body and a_0 and M are the velocity of sound and the Mach number in the undisturbed stream, the two-dimensional potential equation is transformed into

$$[1 - (\gamma - 1)Kf_\xi - \tfrac{1}{2}(\gamma + 1)f_\eta^2]f_{\eta\eta} = K^2 f_{\xi\xi} + 2Kf_\eta f_{\xi\eta}, \tag{2}$$

where

$$K = M\delta/b \tag{3}$$

is a fundamental similarity parameter. The boundary conditions may be expressed

$$f_\xi = f_\eta = 0 \quad \text{at} \quad \xi = -\infty, \tag{4a}$$

$$f_\eta = KH'(\xi) \quad \text{at} \quad \eta = H(\xi), \tag{4b}$$

where $y = H \cdot \delta$ defines the body shape. Since the equation is not linearized it is not permissible to satisfy the shape boundary condition at $\eta = 0$. Two potential flows with similar bodies and the same value of K are thus given by the same mathematical solutions and are similar. The drag and lift coefficients for bodies of similar shape may be expressed

$$C_D = \frac{1}{M^3}\Delta(K) \tag{5a}$$

$$C_L = \frac{1}{M^2}\Lambda(K) \tag{5b}$$

Professor Tsien also demonstrated the analogous law for axially symmetric flow.

It is the purpose of this note to point out that this similarity law is both much simpler in concept and much more general in scope than has been previously indicated; it is, in fact, applicable to three-dimensional flow with shock waves and rotation.

The two-dimensional hypersonic potential equation expressed by Eq. (2) is identical with the one-dimensional non-stationary potential equation in y and t under the transformation

$$t = T\xi \tag{6}$$

with (1b) and (1c), provided the replacement

$$T = b/Ma_0 \tag{7}$$

* Received Jan. 16, 1947.

[1] H. S. Tsien, *Similarity laws of hypersonic flows*, Jour. Math. Phys. **25,** 247–251 (1946).

is made in the definition of K. Also, the boundary conditions (4a), (4b) are the same if now $y = H \cdot \delta$ defines the position of the single boundary as a function of time. This shows that with the slender body hypersonic assumptions, $M \gg 1$, there is no intrinsic dependence upon the axial variable from the point of view of an observer stationary with respect to the undisturbed fluid and the problem becomes a non-stationary one in one fewer space variables. A change of δ with K kept constant is merely a scale transformation in the non-stationary system.

The difference between this point of view and the actual case is that the disturbances at two points on the same streamline on the body are assumed to be in phase. This difference is negligible if the ratio of the signal time between the two points to the time phase difference between the two points is large. Using the fact that in the hypersonic flow over a slender body there are appreciable changes in the velocity of sound but not in the flow velocity, the ratio of these times is equal to the local Mach number, and this parameter being large ensures the validity of the point of view. It is clear that the concept applies to three-dimensional as well as to two-dimensional flow. The presence of shock waves in a flow of extremely high Mach number can change the order of magnitude of the local sound velocity. However, a simple investigation shows that this sound velocity cannot be greater in order of magnitude than $Ma_0\beta$, where β is the inclination of the surface causing the shock and is assumed small but of order $1/M$ or larger. Thus with shocks present, the local Mach number will remain of order β^{-1} or larger. Hence the consideration of the hypersonic flow about a slender body as a non-stationary problem in one less dimension remains valid when shock waves and the resultant entropy changes are present.

The general similitude may be expressed thus: If a slender body of the shape

$$g(\xi, \eta, \zeta) = 0 \tag{8}$$

where

$$x = b\xi, \quad (9a) \qquad y = \delta\eta, \quad (9b) \qquad z = \delta\zeta \quad (9c)$$

is placed in a uniform stream of large Mach number the problem is identical with a non-stationary problem in y, z, and t where

$$t = x/Ma_0 \tag{10}$$

and is characterized by the parameter K as given by Eq. (3). The boundary condition satisfied on the surface is

$$Kg_\xi + \left(\frac{v}{a_0}\right)g_\eta + \left(\frac{w}{a_0}\right)g_\eta = 0 \tag{11}$$

where v and w are the velocity components in the y and z directions, respectively. The drag and lift coefficients based upon an area of magnitude $b\delta$ are given by Eqs. (5a) and (5b).

ON THE SUPERPOSITION OF A HEAT SOURCE AND CONTACT RESISTANCE*

By S. A. SCHAAF (*New York University*)

1. Introduction. The usual condition for a thermal contact resistance[1] is that the temperature discontinuity $T_1 - T_2$ across the interface of two heat conductors [1] and [2] should be proportional to the rate of heat transfer H per unit area there, i.e.

$$T_1 - T_2 = RH, \qquad R = \text{"resistance" constant.} \tag{1}$$

If heat is generated at the same interface, however, the quantity H in (1) cannot be interpreted unless a more specific physical model of the interface is considered. For a

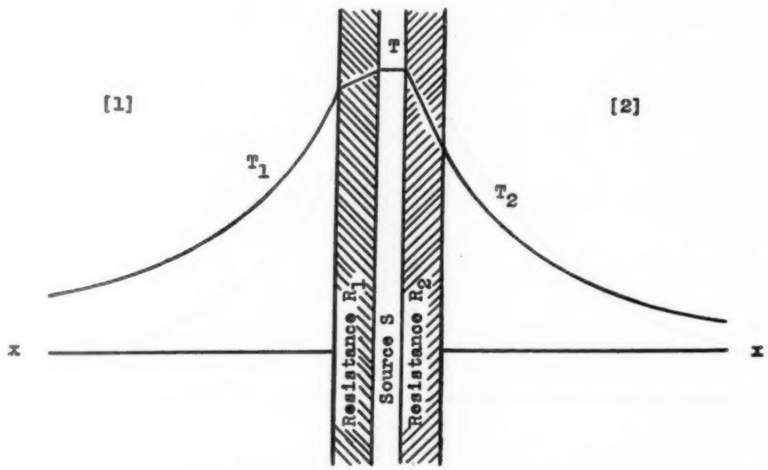

FIG. 1. Magnified view of "lubricated friction" model for interface, consisting in a source S between the two media and separated from them by contact resistances R_1 and R_2. A typical temperature distribution for some fixed time $t>0$ is superimposed.

large class of problems, including those in which the heat source is caused by friction, appropriate physical models of the interface lead to a condition of the form

$$T_1 - T_2 = R(C_1S - H_1) = -R(C_2S - H_2) \qquad C_1 + C_2 = 1, \tag{2}$$

where S is the rate of heat generated per unit area at the interface, H_i is the rate of heat per unit area flowing into the corresponding medium and C_i is a constant whose interpretation depends on the nature of the interface. Condition (2) is obtained by considering models of the interface in which either the contact resistance or the heat source is broken up into two parts. If the source is caused by friction as the two solids slide against each other with the interface as slip plane, these two models correspond respectively to lubricated and to dry friction.

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* Received May 27, 1946.

[1] W. A. Mersman, *Heat conduction in an infinite composite solid with an interface resistance*, Trans. Amer. Math. Soc. **53**, 14–24 (1943).

Lubricated friction. Here we suppose (see Fig. 1) that the heat source is located between two known contact resistances R_1 and R_2, with $R_1+R_2=R$. This is the case if the two solids are separated by a layer of lubricating fluid, for example melted material from one of the solids. Heat is then generated in the turbulent fluid and flows to the two solids through film contact resistances of the ordinary kind. If we suppose that the fluid temperature is T, then according to (1) we have

$$T - T_i = R_i H_i \quad \text{at} \quad x = 0, \quad (i = 1, 2).$$

Eliminating T, using $R_1+R_2=R$ and $H_1+H_2=S$, we obtain

$$T_1 - T_2 = R_2 S - RH_1 = -(R_1 S - RH_2) \quad \text{at} \quad x = 0,$$

which is equivalent to (2) if we put $C_i = 1 - R_i/R$.

Dry friction. An alternative model for the interface (see Fig. 2) is to suppose that

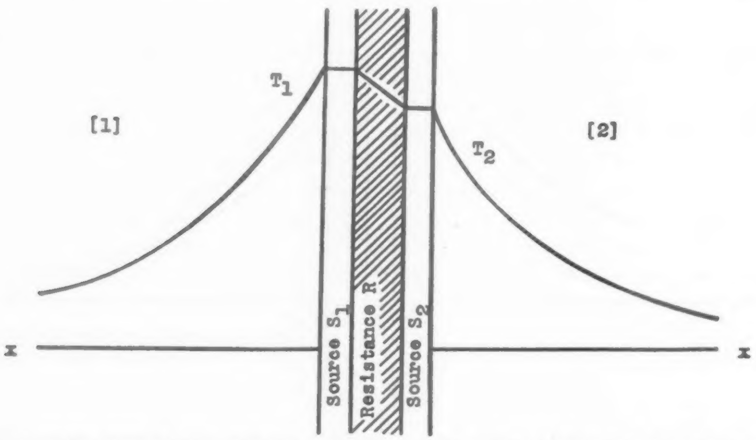

FIG. 2. Magnified view of "dry friction" model for interface, consisting in two sources S_1 and S_2 separated by a contact resistance R, with typical temperature distribution as some fixed time $t>0$.

the heat is generated ***on*** each of the two surfaces, so that there are two sources $S_1(t)$ and $S_2(t)$, with $S_1+S_2=S$, separated by a single contact resistance R. This seems a reasonable model for dry friction, where the heat is generated on the two sliding surfaces which are separated at most points by a very thin air gap which constitutes the contact resistance. It further seems reasonable to suppose that these two sources are proportional, so that

$$S_1(t) = C_1 S(t), \qquad S_2(t) = C_2 S(t), \qquad C_1 + C_2 = 1,$$

where C_i is an empirical constant depending on the elastic and cohesive properties of the two media and on the geometry of the surface roughness, but not on the source strength $S(t)$ nor the temperature. The rate of heat per unit area flowing across the resistance from [1] to [2] is then $S_1(t) - H_1(t)$, so that according to (1) we have

$$T_1 - T_2 = R(S_1 - H_1) = R(H_2 - S_2) \quad \text{at} \quad x = 0 \quad \text{since} \quad S_1 + S_2 = H_1 + H_2.$$

This is equivalent to (2) providing that the assumption $S_i = C_i S$ is valid.

In Sec. 2 we solve a typical heat transfer problem whose formulation involves the boundary condition (2).

2. Problem and solution. Let us consider the case of two semi-infinite media [1] and [2], initially at zero temperature and in contact along the plane $x=0$, where there is a heat source $S(t)$ and a contact resistance R. Then, in the usual notation, we have the following boundary value problem:

$$\frac{\partial^2 T_i}{\partial x^2} = \frac{1}{\alpha_i}\frac{\partial T_i}{\partial t}, \qquad (i = 1, 2), \quad x > 0, \quad t > 0. \tag{3}$$

$$T_i = 0, \qquad t = 0, \qquad x > 0. \tag{4}$$

$$T_1 - T_2 = R\left[C_1 S(t) + K_1\frac{\partial T_1}{\partial x}\right] = -R\left[C_2 S(t) + K_2\frac{\partial T_2}{\partial x}\right], \qquad x = 0, \quad t > 0, \quad C_1 + C_2 = 1 \tag{5}$$

Equation (5) is the same as (2) since

$$H_i = -K_i\frac{\partial T_i}{\partial x} \quad \text{at} \quad x = 0, \quad t > 0. \tag{6}$$

The solution of (3) and (4) in terms of the undetermined functions $H_i(t)$ is well known[2] to be

$$T_i = \beta_i\int_0^t H_i(t-u)\exp(-y_i^2/4u)du/\sqrt{u}, \tag{7}$$

where $y_i = x_i/\sqrt{\alpha_i}$, $\beta_i = \sqrt{\alpha_i}/K_i\sqrt{\pi}$. The functions $H_i(t)$ may then be determined by use of the Laplace transform from the integral equation (5) and the additional requirement

$$H_1(t) + H_2(t) = S(t), \tag{8}$$

which is obtained by equating the two right members in (5) and then using (6). We denote by $f^*(s)$ the transform of $f(t)$, i.e.

$$f^*(s) = \int_0^\infty e^{-st}f(t)dt \tag{9}$$

and recall that

$$\left[\int_0^t f(t-u)g(u)du\right]^* = f^*(s)\cdot g^*(s). \tag{10}$$

Transforming (7), we obtain

$$T_i^* = \beta_i H_i^*(s)\cdot\sqrt{\pi/s}\exp(-y_i\sqrt{s}), \tag{7*}$$

since[3]

$$\left[\frac{\exp(-y_i^2/4t)}{\sqrt{t}}\right]^* = \sqrt{\pi/s}\exp(-y_i\sqrt{s}). \tag{11}$$

[2] H. S. Carslaw, *Conduction of heat in solids*, p. 153.

[3] J. Cossar and A. Erdelyi, *Dictionary of Laplace transforms*, ACS-DSRE No. 71, p. vi–20.

The transforms of equations (5) and (8) are

$$\beta_1 H_1^* \sqrt{\pi/s} - \beta_2 H_2^* \sqrt{\pi/s} = R(C_1 S^* - H_1^*), \tag{5*}$$

$$H_1^* + H_2^* = S^*. \tag{8*}$$

Solving for H_1^*, we obtain

$$H_1^* = \frac{\beta_2\sqrt{\pi} + RC_1\sqrt{s}}{(\beta_1 + \beta_2)\sqrt{\pi} + R\sqrt{s}} S^*. \tag{12}$$

We will treat the cases $S=$constant S_0, and $S=S(t)$ separately. In the former case, we have

$$S^* = S_0/s, \tag{13}$$

and the inverse of (12) is then known[4] to be

$$H_1(t) = \frac{\beta_2 S_0}{\beta_1 + \beta_2} + \frac{S_0(\beta_1 C_1 - \beta_2 C_2)}{\beta_1 + \beta_2} \cdot \exp\left[(\beta_1 + \beta_2)^2 \pi t/R^2\right] \operatorname{erfc}\left[(\beta_1 + \beta_2)\sqrt{\pi t}/R\right], \tag{14}$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du.$$

$H_2(t)$ may be obtained from (14) by interchanging subscripts 1 and 2; the expressions for T_i then follow immediately from (7). We observe that as $t \to \infty$, H_i approaches the constant value it would have for $R=0$, i.e. for no contact resistance. It follows that, although

$$T_i(0, t) \sim \text{const.} \sqrt{t} \quad \text{as} \quad t \to \infty,$$

the temperature discontinuity

$$T_1(0, t) - T_2(0, t) \to \text{const.} \quad \text{as} \quad t \to \infty.$$

For the case $S=S(t)$, we substitute (12) into (7)* to obtain

$$T_1^* = \beta_1 S^* \left[\frac{\beta_2\sqrt{\pi} + C_1 R\sqrt{s}}{(\beta_1 + \beta_2)\sqrt{\pi} + R\sqrt{s}} \cdot \frac{\sqrt{\pi} \exp(-y_1\sqrt{s})}{\sqrt{s}}\right]. \tag{15}$$

This is in the form of a product of two functions with known inverses,[5] so that (10) may be used to obtain

$$T_1 = \beta_1 \int_0^t S(t-u) \cdot \left\{\frac{C_1 \exp(-y_1^2/4u)}{\sqrt{u}} + \frac{\pi(\beta_2 C_2 - \beta_1 C_1)}{R} \right.$$
$$\left. \cdot \exp\left[\frac{\beta_1 + \beta_2}{R} y_1 \sqrt{\pi} + \frac{(\beta_1 + \beta_2)^2}{R^2} \pi u\right] \cdot \operatorname{erfc}\left[\frac{\beta_1 + \beta_2}{R}\sqrt{\pi u} + \frac{y_1}{2\sqrt{u}}\right]\right\} du. \tag{16}$$

The expression for T_2 may be obtained from (16) by interchanging the subscripts 1

[4] J. Cossar and A. Erdelyi, *loc. cit.*, p. vi–76.

[5] J. Cossar and A. Erdelyi, *loc. cit.*, pp. vi–77, 78.

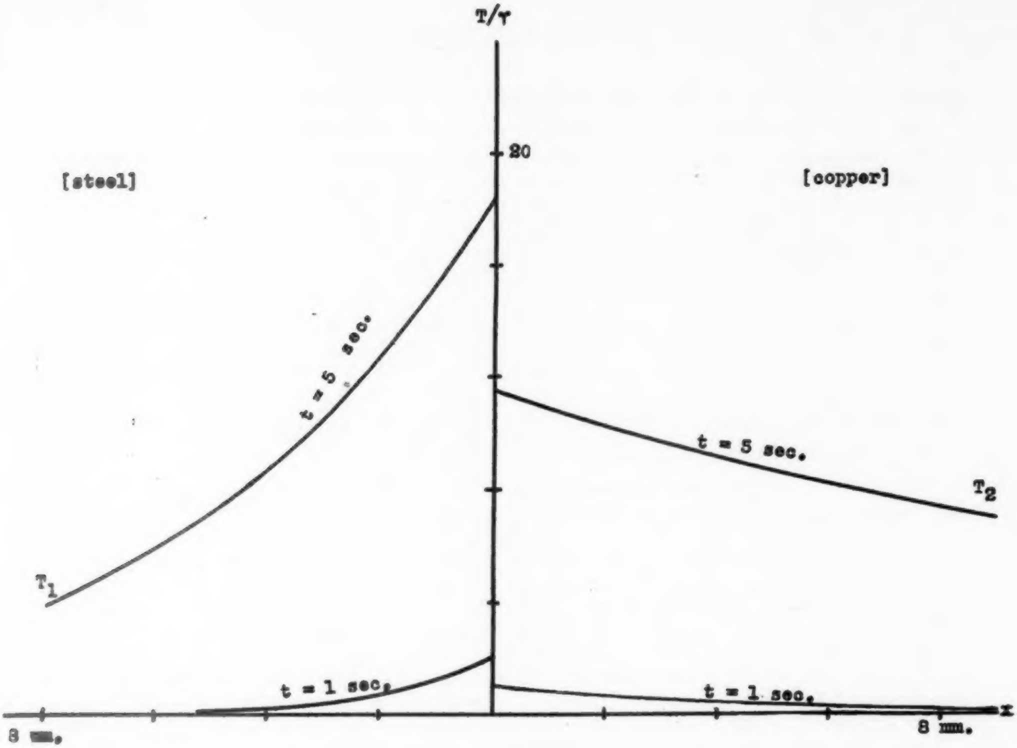

FIG. 3. Temperatures distribution in steel and copper media, separated by a contact resistance and a heat source whose strength is proportional to the time, $S=\gamma t$.

and 2. The evaluation of (16) is given graphically in Fig. 3 for the typical numerical case (in c.g.s.) $\beta_1=1.8$ (steel), $\beta_2=.6$ (copper), $C_1=C_2=1/2$, $R=4.2$, $S(t)\sim t$; which corresponds approximately to the early stages of heat transfer between a steel gun barrel and a copper shell under constant acceleration.

BOOK REVIEWS

Applied mathematics for engineers and physicists. By Louis A. Pipes. McGraw-Hill Book Co., Inc. New York and London, 1946. xiii+618 pp. $5.50.

The mathematical techniques which a beginning graduate student in Engineering should master are outlined and exemplified in this book. The material on such subjects as Laplace transforms, matrices, finite differences, and conformal mapping, is quite complete and occupies a large percentage of the portion of the book which deals with specific problems. An adequate section on special functions is given and the classical equations of mathematical physics are discussed. There appear to be, in fact, only two unfortunate omissions. The calculus of variations is almost entirely neglected and the absence of Sturm-Liouville theory (and hence a discussion of the special functions arising from the wave equation) makes the book definitely less interesting to physicists than to engineers. It should, however, provide an excellent text for a first graduate course in the techniques of applied mathematics.

G. F. CARRIER

Fundamental theory of servomechanisms. By LeRoy A. McColl. D. Van Nostrand Co., Inc., New York, 1945. XVII+130 pp. $2.25.

This book beautifully fulfills its announced purpose of discussing the fundamental theory of servomechanisms. The author succeeds simultaneously in presenting a clear discussion of servomechanisms and in giving the reader a vivid picture of the philosophy of the control problem.

The body of the book consists of eleven chapters, an appendix, and a bibliography. A foreword written by Dr. Warren Weaver sets the stage for the discussion. For the most part, the author discusses linear systems and accordingly makes efficient use of the concepts developed in studying feed-back amplifiers. This point of view combines operational methods with certain straightforward ideas from the theory of complex variables to describe the performance of control system.

Starting with a very simple type of control mechanism, the author builds up the phenomonological and theoretical aspects of the problem, giving in the seventh chapter a detailed analysis of a particular system. The remaining four chapters discuss more specialized topics in linear systems. The appendix discusses some aspects of non-linearity and presents a study of a particular on-off servomechanism.

J. A. KRUMHANSL

Tables of Fractional Powers. Prepared by the Mathematical Tables Project conducted under the sponsorship of the National Bureau of Standards. Official Sponsor: Lyman J. Briggs. Project Director: Arnold N. Lowan. Columbia University Press, New York, 1946. xxx+486 pp. $7.50.

The first part of this useful volume contains tables of the values of A^x for $A=2(1)9$, $x=[0.001(.001)0.01(.01)0.99;\ 15D]$; $A=10$, $x=[0.001(.001)1.000;\ 15D]$; $A=\pi$, $x=(0.001(.001)1.000;\ 15D, 15S]$ and $\pm x=1/4, 1/3, 1/2, 2/3, 3/4$ and $1(1)12$; $A=[0.01(.01)0.99]$, $x=[0.001(.001)0.01(.01)0.99,\ 15D]$ and $A=10^{-3}P$ where P is a prime number between 100 and 1000, $x=[0.001(.001)0.01(.01)0.99;\ 15D]$. The second part contains tables of the values of x^a for $\pm a=1/4, 1/2, 3/4$, $x=[0(.01)9.99;\ 15D]$; $\pm a=1/3, 2/3$, $x=[0(.01)10;\ 15D]$ and $a=[0.01(.01)0.99]$, $x=[0(.01)0.99;\ 7D]$. In the Foreword, F. Bernstein discusses problems the solution of which is facilitated by the use of these tables.

W. PRAGER